The optimal quantity of money over the business cycle and at the zero lower bound

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Abstract

The paper presents a general equilibrium model where agents have limited participation in financial markets and use money to smooth consumption. This framework is consistent with recent empirical findings on money demand. New developments in the heterogeneous agents literature are used to develop a tractable framework with aggregate shocks, where optimal monetary policy can be analyzed. It is shown that the market allocation is not efficient because participating agents do not have the right incentives to save. Monetary policy can restore the first best allocation because of its redistributive effect. When the zero lower bound binds, monetary policy can still restore the first best, but only if the new money is created by open market operations.

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1   Introduction

The rapid expansion of the central bank balance sheets in the US, Japan or in the Euro area after the 2008 crisis has rejuvinated old but deep questions: What are the real effects of money injections? Are these effects, if any, desirable? The answers to these questions obviously depend on the desire of private agents to hold the new money created by monetary authorities, and thus on the evolution of money demand.

The understanding of money demand, and its relationship with financial frictions, has improved recently thanks to some empirical contributions. Recent analysis shows that households' money demand is best understood when one introduces a friction that generates limited participation in financial markets, such as participation costs. In this case, agents use money to smooth consumption between periods at which they adjust their financial portfolio. The distribution of money across households generated by this friction is much more similar to the data than the one generated by alternative money demand (Alvarez and Lippi 2009; Cao et al 2012; Ragot 2014). The initial idea of this foundation for money demand dates back to Baumol and Tobin's seminal contributions, and it has been developed in the limited participation literature in monetary economics.

This paper studies the positive and normative implications of money creation in a model where money demand is based on limited participation. It is shown that this friction generates a new role for monetary policy, both in the business cycle and at the zero lower bound. It is already known that limited participation generates some relevant sort-run effects, such as the liquidity effect of money injection: An increase in the quantity of money decreases the nominal interest rate, as only a part of the population must absorb the new money created (Lucas 1990; Alvarez, Atkeson and Edmond 2009, among others). Nevertheless, this promising literature has faced some difficulties in dealing with agents' heterogeneity (see the literature review below). This has prevented the introduction of additional features which are important to understand the business cycle, such as long-lasting heterogeneity, aggregate sock or capital accumulation. Recent developments in the heterogenous agent literature now allow deriving new results about optimal monetary policy in these environments.

Why is limited participation in financial markets important for monetary economics? Before discussing the model it may be useful to provide some intuitions about the new mechanisms generated by limited participation. First, if agents smooth consumption with money between dates at which they participate in financial markets, the marginal remuneration of saving for many agents is the return on money (roughly the opposite of the inflation rate), the fluctuations of which are in general different from the fluctuation in the marginal productivity of capital. As a consequence, agents do not have the right incentives to save in the business cycle. Second, money creation generates some redistribution across agents because of the heterogeneity in money holdings under limited participation. Through this channel, monetary policy can affect and hopefully improve the incentives to save. Third, at the zero lower bound, even if money and

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interest bearing assets are substitutes, money creation can again generate some redistribution across agents and thus improve the consumption-saving choice of households.

To study these questions, this paper first presents a simple general equilibrium model to derive formal proof. Then, it provides a quantitative framework to study optimal monetary policy with heterogeneous agents and aggregate shocks.

Analyzing the simple model, one first finds that the distortions generated by limited participation are surprisingly not simple. In general capital, accumulation is not optimal as a part of the income generated by capital accumulation is distributed as wages to households, who do not participate in financial markets and thus do not have the right incentives to save. In general equilibrium, this distortion appears as a difference between the social discount factor and the one used by participating agents in their saving choice. The outcome is that capital accumulation after a technology shock is either too high or too low, according to preferences and to the pesistence of the shock. Money creation can restore the first-best allocation by introducing efficient redistribution across agents. For instance, to increase aggregate saving, money creation induces a transfer between non-participating and participating households. This implements the optimal consumption levels.

Second, at the ZLB, the surprising result is that money creation can restore the first best allocation, but only if money is created by open market operations and not by lump-sum transfers. How is it possible that an additional binding constraint does not prevent the implementation of the optimal allocation? At the ZLB, there is an additional money demand by participating agents, who are indifferent between holding money and holding interest-bearing assets. Due to this additional money demand, monetary policy can implement the right incentives to both save and consume, by affecting the inflation rate. At the ZLB, open market operations provide some new money only to participating agents, what generates an amount of redistribution, which is absent when money is created by lump-sum transfers. We first prove this result and then discuss other results in the literature, notably Eggertson and Woodford (2003).

The third Section of the paper presents a quantitative model where households face both idiosyncratic and aggregate shocks, and participate infrequently in financial markets. In this setup, heterogeneity is limited using the tools developed in Challe and Ragot (2014) and Challe, Matheron, Ragot and Rubio-Ramirez (2014). The model reproduces well money and consumption inequalities in the US. The first best cannot be achieved because monetary policy alone can not complete the markets, but it is shown that optimal monetary policy face a

tradeoff between restoring the right incentives to save and increasing consumption inequality. Active monetary policy increases capital accumulation by 10% after a postive technology shock, compared to the one obtained with a passive monetary policy. The capital stock remains roughly 10% lower than its first-best level.

All these results are derived with flexible prices. This assumption is made to identify the key mechanisms in a tractable environment. It is not a statement about the actual functioning of the goods market. The potential new effects generated by nominal frictions are discussed as concluding remarks.

Finally, the general outcome of this model is that monetary policy must affect incentives to invest and thus capital accumulation. In an older literature review (see below), it is interesting to observe that this view of monetary policy is common to both Keynes and Hayek. Modern tools used in this paper, such as mechanism design, provide a rigourous method to make this claim.

The rest of the Introduction is the literature review. Section 2 presents the simple model, where distortions of the market economy and the optimal monetary policy are identifier. Section 3 focuses on optimal monetary policy at the Zero Lower Bound (ZLB). Section 4 presents the general model to quantify the mechanisms. Section 5 is the Conclusion.

1.1   Related Literature

Money demand. A recent literature has derived new results on money demand using cross-sectional information about money holdings. The work of Alvarez and Lippi (2009) shows that models with limited participation in financial markets can reproduce the distribution of money if one enriches the model with a stochastic access to cash and some random participation to the goods market. Ragot (2014) shows that the distribution of money, even defined in a narrow sense, is much more similar to the distribution of financial asset than to the distribution of consumption expenditures. In a quantitative model where households face both a cash-in-advance constraint and a participation cost in financial markets, he shows that limited participation accounts for more than 80% of money demand. Recently Alvarez and Lippi (2013) show that in addition to limited participation, lumpy expenditures is an important feature to reproduce a realistic money demand.

Limited Participation. Limited participation models were first introduced to rationalize the liquidity effect of money injections (Grossman and Weiss (1983) and Rotemberg (1984)). This

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literature has to deal with household heterogeneity. Lucas (1990) and Fuerst (1992) use a family structure: Agents within the family are separated at the beginning of the period and join at the end of the period to pool risk. This outcome does not allow for persistent effects of money shocks, which are shown in this paper to be crucial. Some other tools have been introduced. Alvarez, Atkeson and Edmond (2009) use an overlapping-generation structure. The work of Alvarez and Lippi (2009) focuses on partial equilibrium to derive new results on participation rules (and thus equilibrium environment) when households face a rich stochastic structure. As a consequence, the optimal allocation cannot be studied. Finally, models with limited participation in a non-monetary environment have been recenty used by Kaplan and Violante (2013) to study fiscal policy. They show that such limited participation is necessary to reproduce the effect of fiscal shocks.

Optimal monetary policy and limited participation. An extreme form of limited participation is an environment where there are no financial markets and where households only use money to smooth consumption. These environments are first analyzed in Bewley (1983) and then in a vast literature. Recently Lippi, Ragni and Trachter (2013) wrote one of the few papers studying optimal time-varying monetary policy in such an environment. The role of monetary policy is to improve risk-sharing across agents. Obviously, these environments cannot be used to study distortions in the incentives to accumulate capital. To my knowledge the role of monetary policy to affect the incentive to save, when participation in financial market is limited, has not been identified in the literature.

1.2   Older literature: Keynes and Hayek

It is interesting to note that both the market failure induced by monetary saving, and the role of monetary policy in affecting the incentives to save, was discussed by both Keynes and Hayek. First, as Chamley (2012) and (2013) noticed, the idea that monetary policy and incomplete markets are linked can be found in Keynes, who claim that saving in money generates wrong investment incentives, in his chapter on "Investment Incentives":

> "An individual decision to save does not, in actual fact, involve the placing of any specific forward order for consumption, but merely the cancellation of a present order. For this overlooks the fact that there is always an alternative to the ownership of real capital-assets, namely the ownership of money and debts" (Keynes, 1936).

Monetary saving is identified by Keynes as a potential distortion for the incentive to invest. The fact that saving buying debt can generate the same distortions is also a correct intuition. When debts are inside money, the return on which is different from the marginal productivity of capital, the incentives to save are distorted. This is discussed in Section 2.4.3, devoted to inside liquidity.

Second, the idea that expansionary monetary policy induces capital accumulation was strongly defended by Hayek:

> "The theory that an increase of money brings about an increase of capital, which has recently become very popular under the name of 'forced saving', is even older than the one we have just been considering. [...] An increase in money supply[...] made available to entrepreneurs would cause an increase in the demand for producers' goods in relation to consumers' goods " (Hayek, 1931).

Hayek argued forcefully that monetary policy should be neutral. This is not a claim for a totally inactive monetary policy, but for a monetary policy that does not create distortions. Using modern economic tools the notion of a neutral monetary policy is easy to define: it reduces distortions. The model below shows that a neutral monetary policy is an active one.

2   The Simple model

The simple model allows fully charaterizing all the distortions. Time is discrete and periods are indexed by $t = 0...$ The simple model features a closed economy populated by a continuum of households indexed by i and uniformly distributed along the unit interval, as well as a representative firm. Households have a CRRA period utility function $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$ if $\sigma \neq 1$ and $u(c) = \log(c)$ if $\sigma = 1$. The discount factor is β. It is assumed that the economy is composed of two types of households. There is a fraction Ω of agents, denoted as N-households, who do not participate in financial markets and who can only use money to smooth consumption. The remaining fraction $1 - \Omega$ of households, denoted as $P-$households, participate in financial markets.

It is easy to provide a microfoundation for this limited participation structure with fixed participation costs as in Alvarez et al (2002) for instance. To simplify the exposition, limited participation is now directly assumed, as in Alvarez and Lippi (2014).

2.1 Agents

2.1.1 Non-participating households

N-households are denoted by the upper-script n. A fraction $\Omega/2$ consumes in odd periods and receives labor income in even periods. The other fraction $\Omega/2$ consumes in even periods and receives labor income in odd periods. When working, households supply one unit of labor and get a nominal wage W_t. In all periods, households pay nominal taxes $P_t \tau_t$, where P_t is the price of one unit of final goods and τ_t is taxes in real terms.

When households do not consume, their money demand is their total income $M_t^n = W_t - P_t \tau_t$. From now on, we denote real variables with lowercase. For instance $m_t^n = M_t^n/P_t$. This equality is thus in real terms

$$m_t^n = w_t - \tau_t \tag{1}$$

Households cannot issue money. When households consume, it is guessed that they spend all their money holdings, and the condition for it to be the case is provided below. Denote as c_t^n the consumption of non-participating households in period t:

$$c_t^n = \frac{m_{t-1}^n}{1 + \pi_t} - \tau_t \tag{2}$$

where $\pi_t = P_t/P_{t-1} - 1$ is the net inflation rate. The condition for households not to hold money when they consume is

$$u'(c_t^n) > \beta^2 \frac{1}{1 + \pi_{t+1}} \frac{1}{1 + \pi_{t+2}} u'(c_{t+2}^n)$$

This condition is checked in all examples below.

2.1.2 Participating Households

Variables concerning $P-$households are indicated by the upper-script p. These households supply one unit of labor every period. They participate in money and financial markets, where they buy interest bearing assets[1]. $P-$ households can buy three types of assets: money, government bonds and the capital of firms. In period t, they buy a quantity b_{t+1}^g of government bonds, which pay a nominal interest rate i_t between period t and period $t+1$. They buy a quantity k_{t+1}^p of financial assets, which yield a real return $1 + r_{t+1}$ between period t and $t+1$ and they

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[1]It is direct to introduce consumption every two-periods for $P-$households for them to have the same utility function as non-participating agents, at the cost of more algebra. As a more general model is presented below, we focus here on the simplest case for participating agents.

buy a real quantity m_t^p of money. The budget constraint of a representative $P-$households and the conditions on money holdings are, in real terms:

$$b_{t+1}^g + k_{t+1}^p + m_t^p + c_t^p = w_t - \tau_t + (1 + r_t) k_t^p + \frac{1 + i_{t-1}}{1 + \pi_t} b_t^g + \frac{m_{t-1}^p}{1 + \pi_t} \tag{3}$$

$$m_t^p \geq 0 \tag{4}$$

where c_t^p is real consumption w_t is real labor income, $(1 + r_t) k_t^p$ is the return of financial investment and $\frac{1 + i_{t-1}}{1 + \pi_t} b_t^g$ is the real return on government debt and depends on the the inflation rate π_t. Standard intertemporal utility maximization yields the three Euler equations:

$$u' (c^p) = \beta E_t (1 + r_{t+1}) u' (c_{t+1}^p) \tag{5}$$

$$u' (c^p) = \beta (1 + i_t) E_t \frac{u' (c_{t+1}^p)}{1 + \pi_{t+1}} \tag{6}$$

$$u' (c^p) \geq \beta E_t \frac{u' (c_{t+1}^p)}{1 + \pi_{t+1}} \tag{7}$$

Obviously money is a strictly dominated asset, as long as $i_t > 0$. As a consequence, money may be held by $P-$agents only if $i_t = 0$, i.e. only if ZLB binds.

2.1.3 Firms

There is a unit mass of firms, which produce with capital and labor. Capital must be installed one period before production and is assumed to fully depreciate in production. The production function is Cobb-Douglas $Y_t = A_t K_t^\mu L_t^{1-\mu}$ where K_t, L_t and A_t are respectively the capital stock, the labor hired and the technology level at the beginning of period t. Profit maximization is $\max_{K,L} A_t K^\mu L^{1-\mu} - w_t L - (1 + r_t) K$. It yields the following two first order conditions:

$$w_t = (1 - \mu) A_t K_t^\mu L^{-\mu} \tag{8}$$

$$1 + r_t = \mu A_t K_t^{\mu-1} L^{1-\mu} \tag{9}$$

The level of technology A_t will follow a dynamics specified below. It is assumed that the steady state technology level is defined as $A_t = 1$ and that the agents can form expectations about the next future value of A_{t+1} in each period t.

2.1.4 Budget of the State and Open Market operations

The State raises lump-sum taxes, τ_t. It issues a real quantity b_t of one-period debt and receives the profits of the central banks, which are denoted as Θ_t. This revenue is used to repay the

interest on the debt $\frac{1+i_{t-1}}{1+\pi_t} b_{t-1}$, which depends on the current inflation rate π_t and on the nominal interest rate i_t. To simplify the algebra, we do not assume public spending. The budget constraint of the State is

$$\frac{1 + i_{t-1}}{1 + \pi_t} b_t = b_{t+1} + \tau_t + \Theta_t$$

The State sets taxes such that public debt is constant in each period and equal to $\bar{b}^2$. It simply gives

$$\tau_t = \frac{i_{t-1} - \pi_t}{1 + \pi_t} \bar{b} - \Theta_t \tag{10}$$

To keep the algebra simple, and without loss of generality, we now assume that $\bar{b} = 0$. This assumption is simply normalization of the tax level.

Monetary Policy

It is assumed that the new money is created by open market operations. The difference between this assumption and the simpler process of money creation (such as helicopter drops) is only relevant at the ZLB. To save some space, open market operations are introduced as a benchmark, but we discuss alternative money creation below. The central bank creates a nominal quantity of money M_t^{CB}. The real quantity is $m_t^{CB} = M_t^{CB}/P_t$ and it is used to buy a real quantity b_{t+1}^{CB} of asset by open market operation (to be consistent with the households program, b_{t+1}^{CB} denoted the quantity of bonds bought in period t). Denote as M^{tot} the total nominal quantity of money. The law of motion of M^{tot} is simply $M_t^{tot} = M_{t-1}^{tot} + M_t^{CB}$, or in real terms:

$$m_t^{tot} = \frac{m_{t-1}^{tot}}{1 + \pi_t} + m_t^{CB} \tag{11}$$

The period t real profits of the central bank (which bought a real quantity b_t^{CB} of public debt in period $t-1$) are

$$\Theta_t = \frac{1 + i_{t-1}}{1 + \pi_t} b_t^{CB} \tag{12}$$

It is useful to introduce outstanding public debt, which is the difference between the amount of public debt issued by the state and the amount of financial assets bought by the central bank $b_t^o = b_t - b_t^{CB}$. Using equality (12) and as $m_t^{CB} = b_{t+1}^{CB}$ the consolidated budget constraint of the State is simply $b_{t+1}^o = -m_t^{CB}$. Using (10), taxes can be written as

$$\tau_t = -\frac{1 + i_{t-1}}{1 + \pi_t} m_{t-1}^{CB} \tag{13}$$

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[2] The introduction of alternative fiscal rules is straightforward. As this article does not focus on the interaction between monetary and fiscal policy, we focus on the case of constant public debt.

2.2 Market equilibrium, equilibrium definition and steady state

There are four markets in this economy. First, the equilibrium of the money market is

$$m^{tot} = \frac{\Omega}{2} m_t^n + (1 - \Omega) m_t^p \tag{14}$$

The previous equality stipulates that half of the N-households $(\Omega/2)$ hold money at the end of each period and that P-agents might hold some money. As only P-households participate in financial markets, the equilibria of the bond and capital markets are

$$(1 - \Omega) b_t^g = b_t^o \tag{15}$$

$$(1 - \Omega) k_t^p = K_t \tag{16}$$

The goods market equilibrium is

$$(1 - \Omega) c_t^p + \frac{\Omega c_t^n}{2} - K_{t+1} = Y_t \tag{17}$$

As half N-households and all P-households supply one unit of labor, the labor market equilibrium is $L_t = L$, where

$$L \equiv 1 - \Omega + \frac{\Omega}{2} = 1 - \Omega/2 \tag{18}$$

Given the process for the technology and for a given monetary policy, an equilibrium of this economy is a sequence of individual choices and prices $\{c_t^n, m_t^n, c_t^p, 1 + r_t, 1 + \pi_t, w_t\}$ and a sequence of money stock, outstanding public debt and taxes $\{m_t^{tot}, b_t^o, \tau_t\}$ such that agents make optimal choices, the budget of the State is balanced, public debt is constant and markets clear.

2.2.1 Steady State

Under these assumptions, the steady state of the model gives first insights about equilibrium allocations. Steady state variables are indicated with a star. Assume that there is no money creation $m^{CB*} = 0$, then the steady state inflation rate is $\pi^* = 0$ and the steady state level of taxes is $\tau^* = 0$, from equations (13). The real interest rate is given by the Euler equation of participating agents in steady state. It implies that $1 + r^* = 1/\beta$. One easily deduces the steady state capital stock from equation (9): $K^* = L (\mu\beta)^{\frac{1}{1-\mu}}$. From this expression one finds the steady state consumption level of each participating agent:

$$c^{n*} = m^{n*} = w^* = (1 - \mu) (\mu\beta)^{\frac{\mu}{1-\mu}} \tag{19}$$

$$c^{p*} = \left[\left(\frac{1}{\beta} - 1 \right) \frac{L (\Omega)}{1 - \Omega} + (1 - \mu) \right] (\mu\beta)^{\frac{\mu}{1-\mu}} \tag{20}$$

The consumption of participating agents is obviously higher than the one of non-participating agents, by an amount which is equal to the net gain from financial market participation.

2.3 Optimal allocation and steady state comparison

The optimal allocation is defined as a benchmark to study the distortions of the market economy. Without loss of generality, it is assumed that the planner gives a weight ω_p to $P-$households and a weight 1 to $N-$households. We use the tilde to indicate the optimal allocation. For instance $\hat{c}_t^n$ is the optimal consumption of a N-household in period t. The intertemporal social welfare function is

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{\Omega}{2} u\left(\tilde{c}_t^n\right) + \omega_p \left(1 - \Omega\right) u\left(\tilde{c}_t^p\right) \right) \tag{21}$$

and the resource constraint of the planner is

$$\frac{\Omega \tilde{c}_t^n}{2} + \left(1 - \Omega\right) \tilde{c}_t^p + \tilde{K}_t = A_t \tilde{K}_{t-1}^\mu L^{1-\mu} \tag{22}$$

Solving the program one finds $\tilde{c}_t^n = \omega_p^{-\frac{1}{\sigma}} \tilde{c}_t^p$. In words, the ratio of consumption of participating and non-participating households is constant over the business cycle. With this property the Euler equation is

$$u'\left(\tilde{c}_t^p\right) = \beta E_t \left(1 + \tilde{r}_{t+1}\right) u'\left(\tilde{c}_{t+1}^p\right) \tag{23}$$

where $1 + \tilde{r}_t = \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu}$ is the marginal productivity of capital in the optimal allocation. The resource constraint of the planner is $\tilde{K}_{t+1} + \left(\frac{\Omega}{2} \left(\omega_p\right)^{-\frac{1}{\sigma}} + \left(1 - \Omega\right) \right) \tilde{c}_t^p = A_t \tilde{K}_t^\mu L^{1-\mu}$. This budget constraint and the Euler equation (23) fully characterize the optimal allocation.

First, one can easily compare the market and optimal allocation in steady state, i.e. when there is no money creation $m_t^{CB} = 0$ for all t and where $A = 1$. The following Proposition summarizes the result. All proofs are in the Appendix.

Proposition 1 *1) In the market economy, the steady-state inflation rate π does not affect the capital stock but only the allocation of consumption across agents.*

2) If

$$\omega_p = \left(1 + \frac{\mu}{1 - \mu} \left(1 - \beta\right) \frac{\Omega/2 + 1 - \Omega}{1 - \Omega}\right)^{-\sigma}, \tag{24}$$

then the steady-state market equilibrium is optimal when $\pi = 0$.

The proposition first states that steady-state inflation does not affect the capital stock. This is indeed determined by the discount factor of participating agents, which pins down the real

interest rate. Second, as money is held only by a fraction of the population, money creation generates an inflation tax which is a transfer across agents. As a consequence, for the particular value of a steady-state inflation rate $\pi = 0$, the market allocation of consumption is optimal for the value of the weight ω^p given in the Proposition.

In what follows, it is assumed that ω^p has the value given in the Proposition and that the optimal steady-state inflation rate is thus $\pi = 0$. But it should be clear that all the results below are valid for an arbitrary weight ω^p. Considering the case where the optimal inflation rate is 0 in steady state simplifies the algebra, without loss of generality. Under this assumption, it is possible to focus on business cycle distortions implied by limited participation[3].

2.4   Distortions of the market economy

To further identify the distortions of the market economy, it is now assumed that technology $A_t = e^{a_t}$, where a_t follows an AR(1) process

$$a_t = \rho^a a_{t-1} + \varepsilon_t^a \tag{25}$$

The shock ε_t^a a white noise $\mathcal{N}\left(0, \sigma_a^2\right)$.

This Section first identifies the distortions of the limited participation focusing on the case where there is no money creation $m_t^{CB} = 0$, as a benchmark. The distortions are surprisingly not obvious. As a consequence, we first focus on the case where the ZLB does not bind.

2.4.1   Identifying the distortions

By assumption, the steady-state allocation is optimal. The identification of distortions is thus the analysis of the distortions in the incentives to save of participating agents. Indeed, if participating agents save an optimal amount (i.e equal to the first best defined in the previous section), next-period production is optimal and they consume the optimal amount. In this case, the consumption of non-participating agents is optimal, because of the goods market equilibrium (17).

To understand the distortions of the incentives to save, one can iterate forward the budget constraint of participating agents (using the transversality condition) to derive their intertem-

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[3]Steady-state distortions would be easy to correct with simple transfers between households.

poral budget constraint. This gives

$$\sum_{k=t}^{\infty} \frac{c_k^p}{\prod_{j=t}^{k} [(1+r_j)\,\Theta\,(\Omega)]} = \frac{K_t}{1-\Omega} \tag{26}$$

where $\Theta\,(\Omega) \equiv \frac{1-\mu}{\mu}\frac{1}{1-\frac{\Omega}{2}} + \frac{1}{1-\Omega}$. The expression (26) is derived in Appendix. The right-hand side is the current per capita wealth of $P-$households. The left-hand side is the consumption stream, discounted with the relevant period discount factor, $(1+r_j)\,\Theta\,(\Omega)$. In this discount factor, the term $\Theta\,(\Omega)$ summarizes two distortions of the limited participation economy. First, participating households hold all the capital stock. Second, they obtain only a part of the social return of capital, as a fraction Ω is distributed as wages to non-participating households.

To observe this, one can derive a similar expression for the optimal allocation. Denoting $1 + \tilde{r}_t \equiv \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu}$ as the marginal productivity of capital in the optimal allocation, the solution of the program (21) yields

$$\sum_{k=t}^{\infty} \frac{\tilde{c}_k^p}{\prod_{j=t}^{k} [(1+\tilde{r}_j)\,\Theta\,(0)]} = \frac{1}{\Psi\,(\Omega)} \frac{\tilde{K}_t}{1-\Omega}, \tag{27}$$

where $\Psi\,(\Omega) = 1 + \frac{1}{2}\frac{\Omega}{1-\Omega}\,(\omega_p)^{-\frac{1}{\sigma}}$. The coefficient $\Psi\,(\Omega)$ at the right-hand side depends on the Pareto weight ω_p. It ensures the the market and optimal allocations are the same in their steady state. The left hand side is the discounted sum of consumption, where the period discount factor is $(1+\tilde{r}_j)\,\Theta\,(0)$. One can check that $\Theta\,(\Omega)$ is increasing in Ω. As a consequence, the market discount factor is always higher than the optimal one. The effect of such a distortion is not obvious. It depends on the curvature of the utility function, which captures the relative importance of substitution and income effects. In the log case, it is known that both effects cancel out. The next Section shows that this curvature is indeed crucial for the difference between the optimal and market allocations.

2.4.2 Implication of the distortions

This Section considers a first-order approximation to fully solve the model. It is now assumed that the size of the technology shocks are small enough such that a first-order approximation is relevant. The proportional deviation of the variables x_t to its steady-state value is denoted $\hat{x}_t$, that is $x_t = x^*\,(1 + \hat{x}_t)$. We denote as $\frac{\partial \hat{K}}{\partial \varepsilon^a}$ the increase in the proportional deviation of the

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capital stock on impact after a technology shock for the market economy. The same increase for the first best allocation is denoted as $\frac{\partial \widehat{\widehat{K}}}{\partial \varepsilon^a}$.

The next proposition summarizes the results.

Proposition 2 *Effect of a small technology shock. Assume that $m_t^{CB} = 0$, then*
1) If $\sigma = 1$ then the market and the optimal allocations are the same.
2) $\sigma > 1$ and σ close to 1, then

- *If ρ^a is high then $\frac{\partial \widehat{\widehat{K}}}{\partial \varepsilon^a} < \frac{\partial \hat{K}}{\partial \varepsilon^a}$*

- *If ρ^a is low then $\frac{\partial \widehat{\widehat{K}}}{\partial \varepsilon^a} > \frac{\partial \hat{K}}{\partial \varepsilon^a}$*

The Proposition considers two cases. First, when $\sigma = 1$ the market and optimal allocations are the same, whatever the fraction of people participating in financial markets. In this case and as explained above, income and substitution effects balance each other. As the income of participating agents is equal to its optimal level in steady state, the equilibrium fluctuations of the capital stock are also optimal in the business cycle.

Following the business cycle literature, the case $\sigma > 1$ is here considered as the relevant one. Moreover, it is assumed that σ is close to 1 to derive additional results. The Proposition states that the direction of the distortion in market economies depends on the persistence of the technology shock. For a high persistence, the market economy invests too much on impact compared to the first best allocation ($\frac{\partial \widehat{\widehat{K}}}{\partial \varepsilon^a} < \frac{\partial \hat{K}}{\partial \varepsilon^a}$). For a low persistence, the market economy does not invest enough on impact after a technology shock. When the technology shock is transitory, on impact the central planner increases the capital stock to benefit from the transitory change in productivity and then the capital stock decreases to its steady-state value[4]. In the market economy, as $P-$households do not get all the economic return on investment (as some wages are paid to $N-$households), they do have the incentives to optimally invest. When the technology shock is very persistent, the optimal consumption path of households first increases before it goes back to its steady-state value, because the economy experiences a persistent wealth effect. In the market economy, P-households do not fully perceive this wealth effect. As a consequence, they consume too little, or save too much compared to the optimal allocation.

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[4]More precisely, it is known from the RBC literature that the consumption increases after a very persistent technology shock because of a positive wealth effect. It decreases the period after the transitory technology to benefit from the temporary increase in the productivity of capital.

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Intuitively, the market economy under-reacts to technology shock. Investment is too low to benefit from transitory shocks. It is too high for a permanent shock, as the wealth effect is not fully internalized.

2.4.3 A remark on inside money

The result of this Section does not depend on the formalization of the money supply. In particular, it does not depend on money being outside money. The results would be the same if money were inside money, as long as the return on inside money is different from the marginal productivity of capital. This result is proved in the Technical Appendix to save some space, but the intuition is simple. In general equilibrium what is not consumed must be invested. As a consequence, all the monetary savings (be it outside or inside money) are invested. The key distortion relies on the incentives to save, and thus the return on the money. Frictions generating a liquidity premium on some assets also distort the incentives not to consume.

2.5 Optimal Monetary Policy

Optimal monetary policy is now derived in this environment. We consider that the central bank creates some money in each period observing the state of the economy[5]. As it is shown that the optimal monetary policy implements the first best, there are no commitment issues as the central bank has no incentives to deviate in any period.

To identify the effect of monetary policy in this environment, using the budget constraint (3), together with the equations (13), (15), and (16), the budget constraint of participating households can be written as

$$K_{t+1} + (1 - \Omega) c_t^p = Y_t - w_t \frac{\Omega}{2} + \underbrace{m_t^{CB} - \Omega \frac{1 + i_{t-1}}{1 + \pi_t} m_{t-1}^{CB}}_{\text{implied money transfer}} \tag{28}$$

The previous equality shows that money policy acts as a lumpsum transfer to participating household. (as a general equilibrium effect). One can write the budget constraint of the central

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planner (22) in a similar form:

$$\breve{K}_{t+1} + (1 - \Omega) \breve{c}_t^p = \breve{Y}_t - \breve{w}_t \frac{\Omega}{2} + \underbrace{\frac{\Omega}{2} (\tilde{w}_t - \tilde{c}_t^n)}_{\text{missing saving}} \tag{29}$$

where $\breve{Y}_t$ and $\breve{w}_t \equiv (1 - \mu) A_t \tilde{K}_t^\mu L^{-\mu}$ are respectively the optimal level of output and the marginal productivity of labor in the optimal allocation. In the previous constraint, the time-varying difference between income and consumption of non-participating households appears as a transfer in this budget constraint. This difference is denoted the "missing saving", because it is the part of the income of non-participating agents which is actually invested, in the optimal allocation. As a consequence, if monetary policy is able to implement a transfer to participating agents, which compensates for the "missing saving", it may restore the right incentives to save. The following Proposition shows that this intuition is right.

Proposition 3 *1) if $\Omega < \beta$, an active monetary policy can implement the first best allocation.* *2) The optimal money rule has the following form*

$$m_t^{CB} = H(\Omega, A_t, K_t) + \Omega \frac{1 + i_{t-1}}{1 + \pi_t} m_{t-1}^{CB} \tag{30}$$

where the function H is such that $H(0, A_t, K_t) = 0$ and $H(\Omega, 1, K^*) = 0$.

The first part of the proposition shows that an optimal monetary policy can implement the first best if $\Omega < \beta$. This last condition ensures that Blanchard-Kahn conditions are fulfilled and is always satisfied for realistic parameter values[6]. The exact expression of the optimal policy rule as a function of state variables is given in the proof, because it is not insightful. Instead, the second part of the Proposition presents a simple representation of this rule (which is not written as a function of past state variables because of the term π_t which is a function of m_t^{CB}) The last term at the right-hand side of (30) represents the part of the profits of the central bank which is given to N-households. It captures the redistributive effect of monetary policy for participating households. The optimal monetary policy first cancels this redistributive effect and then creates some money to restore the right incentives to save for P-households. This effect on incentives is captured by the function H. When all households participate in financial markets $\Omega = 0$, the incentives to save are optimal and $H = 0$, as expected. Moreover, in steady state $H(\Omega, 1, K^*) = 0$, as the steady-state allocation is optimal. The time-variation

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[6]When this condition is not fulfiled, money creation tends toward infinity to restore the first best.

in the money created by the central bank reproduces the transfer, which corresponds to the "missing" saving of non-participating households identified in the discussion of equation (29). As a consequence, the consumption saving-choice of participating households is optimal. The consumption of non-participating households is thus also optimal, because of the goods market equilibrium.

One can derive some intuitions for the properties of the optimal monetary policy from Proposition (2). When the persistence of technology shock is low (close to 0), the economy underinvests after a positive technology shock. Optimal monetary policy increases capital accumulation by creating money, which is a transfer from non-participating to participating households. Optimal monetary policy is thus procyclical. When the persistence of technology is high (close to 1), then the market economy accumulates too much capital after a positive technology shock. Optimal monetary policy decreases capital accumulation after a persistent positive technology shock. Optimal monetary policy is thus countercyclical.

Figure 1 provides an example. It is assumed that $\beta = 0.99; \mu = 0.4; \sigma = 3; \Omega = 0.7$ such that 30% of the population participates in financial markets in each period and $\rho^a = 0.95$, which is a standard value. We simulate a persistent *negative* technology shock, to mimic a recession. The figure reproduces the technology shock after a deviation of 1% of TFP, consumption of N-households c^n, the consumption of P-households, c^p, the capital stock K and the money supply which reproduces the first best allocation. For all variables, the solid red line is the market economy when no money is created $m_t^{CB} = 0$. The dashed line is the optimal allocation, which is obtained when $m_t^{CB} = \tilde{m}_t^{CB}$ for the same technology shock. The optimal policy rule is represented as m_t^{CB}. The figure represents the proportional deviation in percent, except for m^{CB} which is level. In the market economy, there is not enough capital accumulation (the capital stock decreases too much after a technology shock). The State increases the capital stock by increasing the money supply. One can check that for a higher persistence of the technology shock $\rho^a = 0.99$ there is over-accumulation of capital in this economy, consistently with Proposition 2. As a consequence, the threshold of the persistence of the technological shock to generate over-accumulation of capital is high.
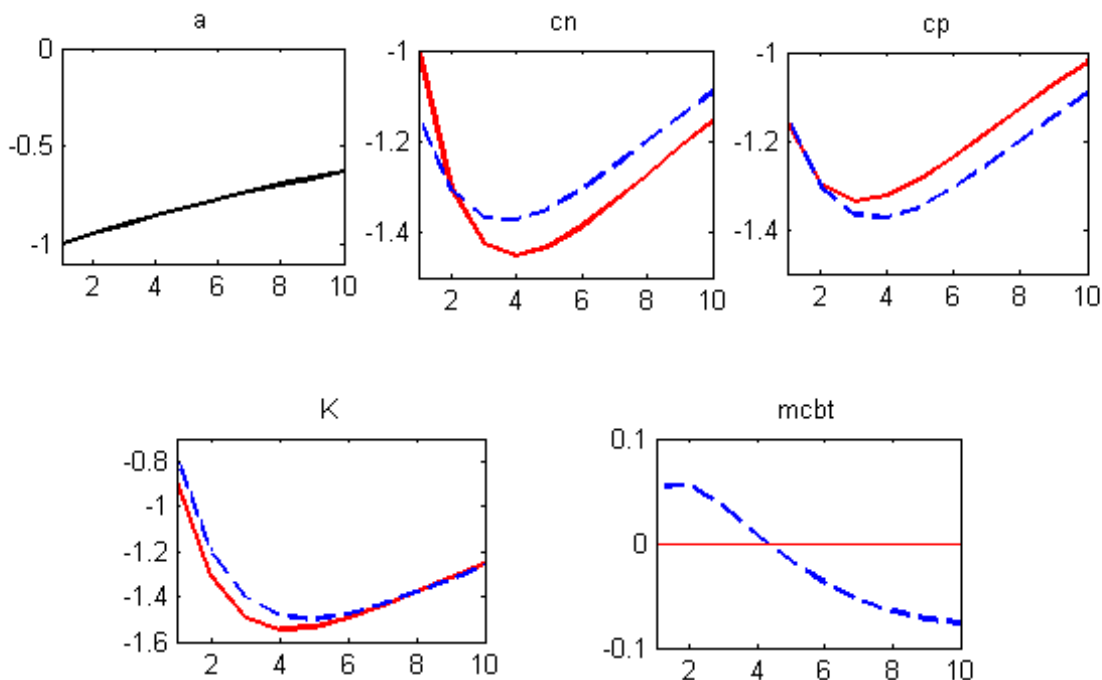
Fig. 1: Optimal monetary policy after a transitory technology shock . The solid line is the market economy with passive monetary policy. The dashed line is the optimal allocation. m^{cb} is the optimal money supply, which reproduces the optimal allocation.

The model is simple enough to be explicitly solved after linearization. It is thus possible to characterize the effect of monetary policy shocks, to study the condition for price determinacy under an interest rate rule or to show that the model exhibits a liquidity effect. As those results are well known in the literature (Alvarez and Lippi, 2014 for instance), they are left for the Technical Appendix.

3 Optimal monetary policy at the Zero Lower Bound

The previous framework has derived the optimal monetary policy assuming that the zero lower bound never binds. A recent literature has studied optimal monetary policy introducing the ZLB as an additional constraint. This simple monetary framework is particularly interesting to study binding ZLB, because the substitution between money and other assets is easy to analyze.

To be consistent with the previous Section and to save some space, it is assumed that the

only shock is a technological shock. This assumption is obviously not a general claim about the nature of the relevant shock for which the ZLB binds[7]. The result of this Section would be obtained for a preference shock, which is often introduced to model the cause of the binding ZLB in models without capital.

Assume that the economy is initially in steady state and that the initial nominal money stock is M, at which the inflation rate is 0. This inflation rate is optimal due to the choice of ω^p. The economy is hit in period 0 by an unexpected sequence of technology level $\{A_t\}_{t=0..\infty}$. The whole sequence of shocks is known at date 0 and the economy is not hit by other shocks at any further date. It is assumed that $lim_{t\to+\infty} A_t = 1$, such that the economy goes back to the steady state.

As before the optimal allocation is defined as the maximization of the objective defined in (21), subject to the budget constraint (22) and the sequence of the technology level $\{A_t\}_{t=0..\infty}$, with the weight ω^p given in Proposition 1. The optimal capital stock solving this problem can be found, using standard dynamic programming methods. It is denoted $\{\tilde{K}_t\}_{t=0..\infty}$. The optimal consumptions of participating and non-participating households are respectively $\{\tilde{c}_t^n\}_{t=0..\infty}$ and $\{\tilde{c}_t^p\}_{t=0..\infty}$.

We now study the ability of monetary policy to reach this allocation. As a first result, one can easily show that monetary policy can achieve the optimal allocation, when the ZLB does not bind.

Proposition 4 *If the ZLB does not bind, the first best allocation can be implemented. The optimal money creation is uniquely defined.*

This Proposition is the equivalent of Proposition 3. The proof is somehow simpler, as the allocation is deterministic after period 0. The interesting case arises when the ZLB binds. One can prove that in this case, the first best allocation can also be implemented.

To see the logic of the argument, assume that the ZLB binds for one period only: The nominal interest rate in period is 0, and the ZLB does not bind afterward. In the market economy in the period 0, we have $i_0 = 0$, hence (as $1 + i_0 = (1 + r_1)(1 + \pi_1)$)

$$1 + r_1 = \frac{1}{1 + \pi_1}$$

To derive the intuition for the implementation result, we have to consider the period 0 and period 1 budget constraint. As the return on money is equal to the return on other assets,

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[7]In particular, it is known that TFP has not decreased in the years after 2008 for which the ZLB is binding.

participating households now hold some money. Their budget constraint can be written as

$$K_1 + (1 - \Omega) c_0^p = Y_0 - w_0 \frac{\Omega}{2} + \underbrace{m_0^{CB} - (1 - \Omega) m_0^p}_{\text{period 0 net money transfer}} \tag{31}$$

In the previous budget constraint, we have gathered all the money terms at the right-hand side and they are labeled "net money transfer". Their Euler equation is

$$u' (c_0^p) = \beta (1 + r_1) u' (c_1^p) = \beta \frac{u' (c_1^p)}{1 + \pi_1}$$

In period 1, $P-$households do not hold money, and they have the return on their past saving :

$$K_2 + (1 - \Omega) c_1^p = Y_1 - w_1 \frac{\Omega}{2} + \underbrace{m_1^{CB} - \Omega \frac{1}{1 + \pi_1} m_0^{CB} + (1 - \Omega) \frac{m_0^p}{1 + \pi_1}}_{\text{period 1 net money transfer}} \tag{32}$$

For period 2 onward, the budget constraint is given by (28). Assume that the quantity of money created in period $1, 2, ...$ (for which the ZLB does not bind) implements an optimal transfer to participating households.

To prove that the first best can be implemented, we have to prove that it is possible to satisfy two conditions in period 0. First, monetary policy must implement a transfer to participating households, which corresponds to the "missing saving" defined in (29). Second, the inflation rate between period 0 and period 1 must be equal to the inverse of the optimal marginal return on capital, for the optimal allocation to solve for the Euler equation of participating households.

$$1 + \pi_1 = (1 + r_1)^{-1} = \left(\mu * A_1 \tilde{K}_1^{\mu-1} L^{1-\mu} \right)^{-1} \tag{33}$$

The two equations (32) and (33) form a well-defined system in two unknowns, m_0^P and m_0^{CB} (once the money market equilibrium is used to express π_1 as a function of m_0^P and m_0^{CB}). If this system as a positive solution, then there exists a money supply for which the optimal allocation is a market equilibrium, even when the ZLB bindsh.

This system can have a solution because of the new variable, the quantity of money held by participating households m_0^p, which can be positive only at the ZLB. The next proposition summarizes the result, providing the conditions for the solution to be positive. The proof is in the Appendix.

Proposition 5 *When the ZLB binds for one period, the optimal allocation can be implemented by a unique monetary policy if $m_0^p \geq 0$, where*

$$m_0^p = \frac{1}{1 - \Omega} \left(\left(\left(\frac{1 - \mu}{\mu} \frac{\tilde{K}_1}{L} - \frac{2}{\Omega} \frac{\Delta_1}{\mu A_1} \left(\frac{\tilde{K}_1}{L} \right)^{1-\mu} \right) - \left((1 - \mu) A_0 \tilde{K}_0^\mu L^{-\mu} + \Delta_0 \right) \right) \right)$$

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and

$$\Delta_t \equiv \tilde{K}_{t+1} + (1 - \Omega)\, \tilde{c}_t^p - A_t \tilde{K}_t^\mu L^{1-\mu} + \frac{\Omega}{2}\, (1 - \mu)\, \tilde{K}_t^\mu L^{-\mu},\ t = 0, 1 \tag{34}$$

In the proposition the value of m_0^p is given as a function of the first best allocation $\{\tilde{K}_t\}_{t=0..\infty}$. The two variables Δ_t for $t = 0, 1$ are the missing saving of non-participating households in periods 0 and 1 as a function of the optimal allocation. The economic meaning is that participating households must want to hold some money when the ZLB binds and when the central bank implement the optimal incentive to save. If participating households anticipate a rapid increase in revenue, they may want to issue some money to transfer some wealth from period 1 to period 0. One can consider this case as being more the exception than the rule.

A numerical example. As a simple numerical example, it is assumed that $\sigma = 3$, $\beta = 0.9999$, $\Omega = 0.5$, $\mu = 0.36$. The value of β is close to 1 for the real interest rate to be low, when the shock hits. The nominal interest rate is at 0 for one period. We assume that the shock technology is an AR(1) (in log) defined by $u_0 = -0.02$ and $u_t = \rho^a u_{t-1}$ for $t \geq 1$, where $\rho^a = 0.7$. $A_t = e^{u_t}$
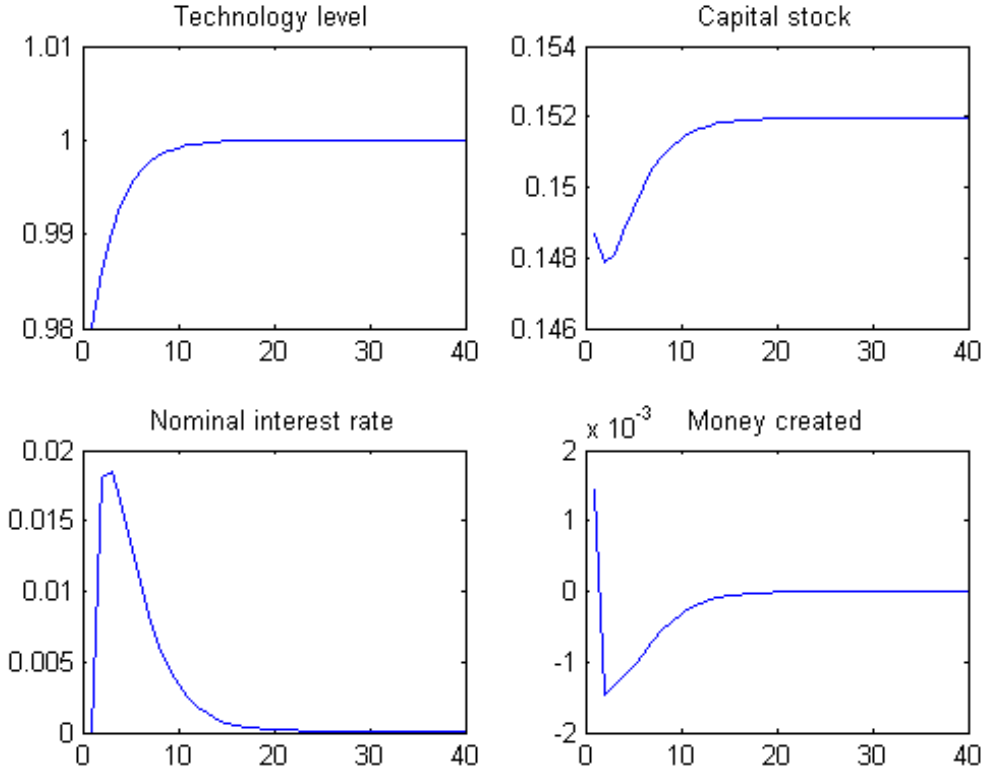
Fig. 1 : Optimal monetary policy when the ZLB binds for one period. The four variables A_t, K_t, i_t and m_t^{CB} are represented as deviations to their steady state values.

It may be interesting to relate the result of this Section with the one of Eggertsson and Woodford (2003). These authors consider an environment with the following features: A representative agent, complete financial market, sticky nominal price setting à la Calvo in an economy without capital. They find that money creation at the zero lower bound does not affect aggregate output, nor the price dynamics. The key differences between their economy and the one of this paper is, first, the representation of heterogeneity. As we have limited participation, Ricardian equivalence does not hold and the pure neutrality result of Eggertsson and Woodford (2003) fails to apply. Second and more importantly, this paper introduced explicitly money demand and limited participation. The ZLB is not only a constraint on the nominal interest rate. When it binds, it also implies that money and other assets are substitutes for participation agents, what affects money demand. It is the inclusion of this second effect which drives the results of Proposition 5.

At a more general level, the claim of Eggertsson and Woodford (2003) is that money creation at the ZLB does not change inflation expectations. Although the identification of this effect

is not simple for recent episodes of money injections at the ZLB, Krishnamurthy and Vissing-Jorgensen (2011) for the US and Joyce et al. (2011) for the UK present some evidence that inflation expectations were increased after episodes of quantitative easing both in the US and the UK. Money creation seems thus to affect inflation expectations and thus the real interest rate.

This Section has considered the case where the ZLB binds for one period. It is possible to derive the optimal monetary policy to implement the first best when the ZLB binds after T periods. During these T periods, participating households demand a quantity of money m_t^p, $t = 0..T$. The condition for the monetary policy to implement the first best is $m_t^p \geq 0$, $t = 0..T$. It is not insightful to analytically characterize the conditions under which it is effectively the case. For this reason, we present the construction of monetary policy for this general case in the Technical Appendix.

3.1   The case of lumpsum transfers

The previous Sections proved that monetary policy can reach an optimal allocation even when ZLB is binding, under the maintained assumption that money is created by open market operations. The next Proposition states that this allocation cannot be reached by lumpsum money transfers to all agents.

The next Proposition presents the result.

Proposition 6 *When money is created by lumpsum transfers, the first best allocation cannot be reached when the ZLB binds.*

The proof is in the Appendix. The reason for this result is that when money is distributed by lumpsum transfers to all agents at the ZLB, they can reduce their money demand by exactly the same amount. The equilibria on the money market and on all markets are unchanged. This result depends on all agents holding money and thus that the ZLB is binding (otherwise some agents not holding money could not reduce their money demand which is 0). Open market operations create an additional redistributive effect because the new money created in period t first affects the budget constraint of participating households in the period t (as can be seen in equation (28)), and then affects the budget of non-particiapating households in the following period, due to the redistribution of the profits of the central bank. This new result is a rationale for open market operations for money creation (at least at the ZLB).

The simple model of this Section shows that monetary authorities can generate the optimal incentive to save. One may legitimately ask wether this strong result is not the result of the various simplifying assumptions. To answer this question, a more general level is now considered. Although the model could obviously be generalized in various dimensions, the next Section generalizes the model to match a simplified empirical distribution of money across households. Indeed, as the key mechanism is the redistributive effect of monetary policy in the business cycle, it is important to first investigate the effect of a realistic heterogeneity.

4 The general model

In the previous theoretical model, non-participating households hold the money stock, whereas participating ones do not hold any money. This is not a consistent description of the data, as many households hold both money and financial assets. In this Section a more elaborated model is introduced to study optimal monetary policy, when a realistic heterogeneity across households is introduced.

More precisely, in US data and using the Survey of Consumer Finance (SCF), three groups of households can be identified according to their money and financial holdings. The definition of money used is the sum of checking and saving accounts. A broader definition of money does not change the overall properties of the money distributions. First, around 10% of the population does not hold a checking account. As Bricker et al. (2012) write, these households are disproportionately more likely to have income in the lowest income quintile, to have net worth in the lowest quartile, or to be headed by a person who is not working. Although the quantity of money the household hold is not precisely measured, it can be assumed to be a trivial quantity.

The second group of the population includes the households who have a higher income, who hold some money, but who do not participate in financial markets. These households are roughly in the 2nd to 5th decile of the income distribution. Indeed, one observes that roughly 50% of the US population does not participate in the stock market either directly or indirectly (Bricker et al., 2012). Participation in the bond market (for any type of bonds) is even more concentrated, as roughly 20% of the US population participates in this market.

The third group of households are the ones above the 5th decile of the income distribution. These households hold both money and financial assets. Although the ratio of money over total

wealth decreases with wealth (Erosa and Ventura, 2002), the quantity of money held by these households is much larger than that held by the other groups. In addition, although these agents hold some financial assets, it is known that they participate infrequently in financial markets (Vissing-Jorgensen, 2002; Alvarez 2009).

To summarize, the population is composed of 10% of non-money holders, who are denoted as NM-households. It is composed of 40% of households who only hold money, denoted as non-participating households or $N-$households. Finally, it is composed of 50% of (infrequently) participating households, who are denoted as participating agents or $P-$households.

In addition to limited participation in financial markets, households will face an unemployment risk and may use money to self-insure. This risk will be calibrated on the US labor market. It will be assumed that this risk affects both P and N households, but that NM households have a low but stable income.

Households thus face two types of financial frictions. The first one is limited participation in financial markets, the second one is incomplete insurance for the income risks. The first financial imperfection generateq a money demand in the Baumol-Tobin tradition, whereas the second generates a money demand in the Bewley tradition. Ragot (2014) shows that both risks are quantitatively necessary to generate a realistic money distribution.

Incomplete insurance markets and limited participation models are known to be very difficult to analyze with aggregate shocks. To my knowledge, simulation techniques do not allow to study such environments in the general case and with aggregate shocks[8]. To capture the essence of limited participation and incomplete markets and to be able to define an optimal monetary policy with aggregate shocks, I use modeling strategy of Challe and Ragot (2013). This strategy can be thought of as an extension of Lucas (1990), who introduces perfect insurance within families. It is assumed that there is perfect insurance within some groups of the population living on "Islands", but that there is no insurance across islands. Households may move randomly across islands. The timing of market opening is then designed such that the model generates an Euler equation for each household, which is consistent with results in the incomplete insurance market literature, but where the heterogeneity is limited to a finite number of household types. It is thus not necessary to follow a continuous distribution of agents as in Krusell and Smith (1998). In this setup, optimal policy with aggregate shocks can easily be studied[9].

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[8]Ragot, (2014) and Kaplan and Violante (2014) study this environment without aggregate shocks.

[9]Heathcote, Storesletten and Violante (2014) use the same modeling trick in a model based on Constantinides and Duffie (1996), where idiosyncratic shocks are persistent. This trick is used here in a Bewley environment,

4.1 Households

All households have the same period utility function u, and have the same discount factor β. They pay lumpsum taxes denoted as τ_t. Households are in different islands, but they all go the same island to consume, work and sell their money. I first present the program and then derive all the Euler equations.

4.1.1 NM- Households

There is a mass Ω^{NM} of NM-households, who have a low and stable income denoted as δ^{NM}. They do not have access to financial markets but they can hold money. These households live on an NM-island and do not move in or out of this island. Denote as $V^{NM}\left(m_t^{NM}\right)$ the intertemporal welfare in period t of a household holding a real quantity of money m_t^{NM} (I keep the time subscript in the whole Section for the sake of clarity). This function solves the Bellman equation

$$V^{NM}\left(m_t^{NM}\right) = \max_{c,m_{t+1}} u\left(c_t^{NM}\right) + \beta EV\left(m_{t+1}^{NM}\right)$$

$$s.t. \qquad c_t^{NM} + m_{t+1}^{NM} = \delta^{NM} + \frac{m_t^{NM}}{1+\pi_t} - \tau_t \tag{35}$$

$$m_{t+1}^{NM} \geq 0$$

The second equation is the budget constraint. The third condition stipulates that households cannot issue money. The solution to this program is simply $m_{t+1}^{NM} = 0$ as long as $u\left(c_t^{NM}\right) > \beta E_t u\left(c_{t+1}^{NM}\right)/\Pi_{t+1}$.

4.1.2 N- Households

There is a mass Ω^N of $N-$households. These households do not participate in financial markets, but they face an employment risk, against which they self-insure. Following the litterature on uninsurable risk, it is assumed that $NM-$households can be either employed or unemployed. When employed, they receive an after-tax labor income $w_t^{NP} - \tau_t$. An employed household stays employed next period with a probability α (and falls into unemployment with a probabiltiy $1-\alpha$). When unemployed, households stay unemployed with a probability ρ (and find work

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where idiosyncratic shocks can be transitory.

with a probability $1 - \rho$). In other words, the transition matrix for the labor risks is

$$
\begin{bmatrix}
\alpha & 1 - \alpha \\
1 - \rho & \rho
\end{bmatrix}
$$

As this transition matrix is not time-varying, the constant fraction of employed households among NP-households is

$$
n = \frac{1 - \rho}{2 - \alpha - \rho} \tag{36}
$$

and the unemployment rate is $1 - n$.

Insurance structure. It is assumed that $N-$households belong to a family, but the family has two locations. Employed households live on an island, denoted as E island, where there is full risk sharing within the island. Unemployed agents live on an island, denoted as U island, where there is full risk-sharing within the island. By the law of large numbers there is a mass $n\Omega^N$ of households in the E island and a mass $(1 - n)\Omega^N$ in the U island. Households who lose their job (with a probability $1 - \alpha$) must travel from the E to the U island at the end of the period, after the consumption-saving choice has been made. Households finding a job (with a probability $1 - \rho$) have to travel from the unemployed to the employed island at the end of the period, after the consumption-saving choice has been made. In each island, the consumption-saving choice is made by a representative of the family head, who maximizes the welfare of the the family. Finally, all households traveling across islands take their money with them.

E **island** All employed $N-$households in the E island supply one unit of labor and earn an after-tax real wage $w_t - \tau_t$. The sequence of actions is the following. First, at the beginning of each period, the family head pools the resources. Second, the aggregate state is revealed, which is the technology shocks and the money creation. Third, the consumption-saving choice is made. Fourth, households' idiosyncratic shock is revealed, and households losing their job travel across islands carrying their money with them.

Denote as m_t^{NE} the per capita real money holdings of employed $N-$households at the beginning of the period after resources are pooled. There is a number $n\Omega^N$ of such agents. Similarly, denote as m_t^{NU} the beginning-of-period per capita money holdings of $N-$households in the U island. There is a number $(1 - n)\Omega^N$ of such households. Denote as $V^{NE}\left(m_t^{NE}\right)$ the intertemporal welfare of an employed household and $V^{NU}\left(m_t^{NU}\right)$ the intertemporal welfare of an unemployed household.

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The program of the family head in the E island is to maximize the utility of all employed households taking into account the consequence of its choices for the whole family of $N-$households. Denote $n\Omega^N V^{NE}\left(m_t^{NE}\right)$ as the maximum intertemporal welfare that the family head can achieve in the E island, with a per capita quantity of money m_t^{NE}. The family head chooses per capita consumption c_t^{NE} and per capita money holdings $\tilde{m}_{t+1}^{NE}$ to solve

$$n\Omega^N V^{NE}\left(m_t^{NE}\right) = \max_{c_t^{NE}, m_{t+1}^{NE}} n\Omega^N u\left(c_t^{NE}\right)$$

$$+ \beta E_t\left[n\Omega^N V^{NE}\left(m_{t+1}^{NE}\right) + (1-n)\Omega^N V^{NU}\left(m_{t+1}^{NU}\right)\right]$$

subject to the per capita budget and the non-negativity money constraint:

$$c_t^{NE} + \tilde{m}_{t+1}^{NE} = \frac{m_t^{NE}}{1+\pi_t} + w_t - \tau_t \tag{37}$$

$$\tilde{m}_{t+1}^{NE} \geq 0$$

$\tilde{m}_{t+1}^{NE}$ is the quantity of money held by each employed agent before some of them have to leave the island. As a consequence, it is different from the next period per capita quantity of money m_{t+1}^{NE} to be determined below.

U **island** All households in the U island are unemployed $N-$households, by construction. They get a per capita home production δ^N. The timing in the U island is the same as in the E island, and these households pay the same taxes τ_t. The representative of the family head chooses the per capita consumption c_t^{NU} and the per capita money holdings $\tilde{m}_{t+1}^{NU}$ to maximize the utility of unemployed households taking into account the effect on all the family. It solves

$$(1-n)\Omega^N V^{NU}\left(m_t^{NU}\right) = \max_{c_t^{NU}, \tilde{m}_{t+1}^{NU}} (1-n)\Omega^N u\left(c_t^{NU}\right)$$

$$+ \beta E_t\left[n\Omega^N V^{NE}\left(m_{t+1}^{NE}\right) + (1-n)\Omega^N V^{NU}\left(m_{t+1}^{NU}\right)\right]$$

subject to the per capita budget and the non-negativity money constraint:

$$c_t^{NU} + \tilde{m}_{t+1}^{NU} = \frac{m_t^{NU}}{1+\pi_t} + \delta^N - \tau_t \tag{38}$$

$$\tilde{m}_{t+1}^{NU} \geq 0$$

4.1.3 P- households

There is a fraction Ω^P of $P-$households. These households face the same employment risk as $NP-$households, with the transition probabilities α and ρ. These households are more productive than $N-$households, and the labor supply is equivalent to κ units of labor of $N-$households.

The wage they recieve when employed is thus κw_t. When unemployed it is assumed that they get a revenue from home production equal to $\kappa \delta^N$.

In addition, these households participate infrequently in financial markets. It is assumed that when they participate, in period t the probability that they participate in period $t + 1$ is α^f (and the probability that they do not participate is $1 - \alpha^f$). When they do not participate in period t the probabiltiy that they do not participate in period $t+1$ is ρ^f (and the probability that they participate is $1 - \rho^f$).

To keep the model tractable, it is assumed that $P-$households can be in three locations or "islands". All $P-$households participating in financial markets are on the same island, denoted as island A. In this island, the family head pools resources and has access to the financial portfolio of the $P-$households.

$P-$households who do not participate in financial markets can be in two other islands. If they are employed, they must travel to the PE island. If they are unemployed, they must travel to the PU island. In all islands, there is a family head who maximizes the welfare of all members of all $P-$households, whatever their location.

The flows across islands are the following. An employed $P-$household who participates in financial markets stays on the same island with a probability α^f. He or she moves to the PE island with a probability $\alpha \left(1 - \alpha^f\right)$, and he or she moves to the PU island with a probability $(1 - \alpha)\left(1 - \alpha^f\right)$. Households in PU island stay in the PU island with a probability $\rho^f \rho$. They move to the A island with probability $1 - \rho^f$, and they move to the PE island with a probablity $\rho^f (1 - \rho)$. Finally, households in the PE island stay in the PE island with a probability $\rho^f \alpha$. They move to the PU island with a probability $\rho^f (1 - \alpha)$. They move the the A island with a probability $1 - \rho^f$.

From the transition matrices, the stationary fraction of $P-$households in island A is $n^A = \left(1 - \rho^f\right) / \left(2 - \alpha^f - \rho^f\right)$. The fraction in $PE-$island is $n^{PE} = \left(1 - n^A\right) n$, where n is determined in equation (36). The fraction of P-households in island PU is $n^{PU} = \left(1 - n^A\right)(1 - n)$. We now describe the program of the family head in each island.

A **island** The family head has access to financial markets. The financial portfolio of P-households includes the (beginning of period) per capita capital stock k_t^A and the per capita level of public debt b_t^A and per capita real quantity of money m_t^A. Denote as $V^A \left(k_t^A, b_t^A, m_t^A\right)$ the intertemporal per capita welfare in the A island. Denote as $V^{PE} \left(m_t^{PE}\right)$ the intertemporal per capita welfare in island PE: It only depends on the per capita quantity of money held

by agents in this island, because the representative of the family head does not participate in financial markets. Finally, denote as $V^{PU}\left(m_t^{PU}\right)$ the intertemporal welfare of agents in the PU island: It only depends on the beginning of period quantity of money held by agents in the PU island.

The family head in the A island maximizes the intertemporal welfare of the households on this island considering the consequence of its choice on the welfare of households moving to other islands. The program of the family head is

$$n^A\Omega^P V^A\left(k_t^A, b_t^A, m_t^A\right) = \max_{c_t^A, \tilde{k}_{t+1}^A, \tilde{b}_{t+1}^A, \tilde{m}_{t+1}^A} n^A\Omega^P u\left(c_t^A\right)$$

$$+ \beta E_t\left[n^A\Omega^P V^A\left(k_{t+1}^A, b_{t+1}^A, m_{t+1}^A\right) + n^{PE}\Omega^P V^{PE}\left(m_{t+1}^{PE}\right) + n^{PU}\Omega^P V^{PU}\left(m_{t+1}^{PU}\right)\right]$$

The family head pools all the resources of households participating in financial markets. As the fraction of employed households is n (and the fraction of unemployed households is $1 - n$), one finds that the resource constraint in per capita terms is

$$c_t^A + \tilde{k}_{t+1}^A + \tilde{b}_{t+1}^A + \tilde{m}_{t+1}^A = \kappa\left(nw_t + (1-n)\,\delta^N\right) - \tau_t + (1+r_t)\,k_t^A + \frac{1+i_{t-1}}{1+\pi_t}b_t^A + \frac{m_t^A}{1+\pi_t}, \quad (39)$$

$$\tilde{m}_{t+1}^A \geq 0$$

where the per capita amount saved in money, bonds and capital (before transitions across islands) is denoted $\tilde{m}_{t+1}^A$, $\tilde{b}_{t+1}^A$ and $\tilde{k}_{t+1}^A$ respectively.

PE **island** On the PE island, the representative of the family head maximizes the intertemporal welfare of employed households who do not participate in financial markets, while considering the consequence of its choice on the welfare of the whole family. The program of the family head is

$$n^{PE}\Omega^P V^{PE}\left(m_t^{PE}\right) = n^{PE}\Omega^P u\left(c_t^{PE}\right) -$$

$$+ \beta E_t\left[n^A\Omega^P V^P\left(k_{t+1}^P, b_{t+1}^P, m_{t+1}^P\right) + n^{PE}\Omega^P V^{PE}\left(m_{t+1}^{PE}\right) + n^{PU}\Omega^P V^{PU}\left(m_{t+1}^{PU}\right)\right]$$

The per capita resource constraint is

$$\tilde{m}_t^{PE} + c_t^{PE} = \kappa w_t - \tau_t + \frac{m_t^{PE}}{1+\pi_t} \tag{40}$$

$$\tilde{m}_t^{PE} \geq 0$$

PU **island**   Following the same steps, the program of the family head on the *PU* island is

$$n^{PU}V^{PU}\left(m_t^{PU}\right) = \left(1-n^A\right)\left(1-n\right)u\left(c_t^{PU}\right)$$
$$+ \beta E_t\left[n^A V^A\left(k_{t+1}^A, b_{t+1}^A, m_{t+1}^A\right) + \left(1-n^A\right)nV^{PE}\left(m_{t+1}^{PE}\right) + \left(1-n^A\right)\left(1-n\right)V^{PU}\left(m_{t+1}^{PU}\right)\right]$$

with the constraint

$$\tilde{m}_t^{PU} + c_t^{PU} = \kappa\delta - \tau_t + \frac{m_t^{PU}}{1+\pi_t} \tag{41}$$
$$\tilde{m}_t^{PU} \geq 0$$

4.1.4   Money holdings and equilibrium structure

The equilibrium is constructed with a guess-and-verify strategy. Indeed, many households choose not to hold money because the return on money is too low. More specifically, we make the following conjecture.

Conjecture : Households in NM, PU, PE island do not hold money, i.e.

$$m_t^{NM} = \tilde{m}_t^{U} = \tilde{m}_t^{PU} = \tilde{m}_t^{PE} = 0 \tag{42}$$

and only households in the E and PE island hold money $\tilde{m}_t^E > 0$, and $\tilde{m}_t^{PE} > 0$.

Implications for money holdings

At each end of period, some agents transit between islands. A size $(1-\alpha)n\Omega^N$ of households travel from island E to island U and the remaining size $\alpha n\Omega^N$ stay in island E. As only households in the E island hold money, the beginning-of-period money held by households in the E island is the quantity of money which stays in this island. As a consequence, the per capita beginning-of-period quantity of money in the E island is $m_{t+1}^{NE} = \alpha n\Omega^N\tilde{m}_{t+1}^{NE}/\left(n\Omega^N\right)$. Along the same line, the per capita beginning-of-period quantity of money in the U island is $m_{t+1}^{NU} = (1-\alpha)n\Omega^N\tilde{m}_t^{NE}/\left((1-n)\Omega^N\right)$. One easily finds,

$$m_{t+1}^{NE} = \alpha\tilde{m}_t^{NE} \text{ and } m_{t+1}^{NU} = (1-\rho)\tilde{m}_{t+1}^{NE}$$

Considering P households, $\left(1-\alpha^f\right)n^A\Omega^P$ households leave the A island with only their money holdings. Among these agents, there is a fraction n who stay employed with a probability α and a fraction $1-n$ who become employed with a probability $1-\rho$. As a consequence, the number of $P-$households moving from island A to island PE is $(\alpha n + (1-\rho)(1-n))*$ $\left(1-\alpha^f\right)n^A\Omega^P = n\left(1-\alpha^f\right)n^A\Omega^P$. As agents in the PE and PU island do not save in money,

the only money held by agents in the PE island is the money brought by agents transiting from the A to the PE island. As a consequence, the per capita beginning-of-period amount of money held by agents in the PE island is

$$m_t^{PE} = \frac{n \left(1 - \alpha^f\right) n^A}{n^{PE}} \tilde{m}_t^A$$

Following the same reasoning, one finds that the per capita beginning-of-period amount of money held by households in the PU island is

$$m_t^{PU} = \frac{(1 - n) \left(1 - \alpha^f\right) n^A}{n^{PU}} \tilde{m}_t^A$$

The beginning-of-period real per capita quantity of money held by agents in the A island is $m_{t+1}^A = \alpha^f \tilde{m}_{t+1}^A$. Finally, as only agents in the A island have access to the portfolio of a participating family, we have $\tilde{k}_{t+1}^A = k_{t+1}^A$ and $b_{t+1}^A = \tilde{b}_{t+1}^A$.

Equilibrium conditions. It is now possible to write the conditions under which the conjecture (42) is satisfied. We show in the Appendix that these conditions are

$$u\left(c_t^{NM}\right) > \beta E_t \frac{u\left(c_t^{NM}\right)}{1 + \pi_{t+1}} \tag{43}$$

$$u'\left(c_t^U\right) > \beta E_t \left[(1 - \rho)\, u'\left(c_{t+1}^E\right) + \rho u'\left(c_{t+1}^U\right)\right] \frac{1}{1 + \pi_{t+1}} \tag{44}$$

$$u'\left(c_t^{PE}\right) > \beta E_t \left[\left(1 - \rho^f\right) u'\left(c_t^A\right) + \rho^f \alpha u'\left(c_t^{PE}\right) + u\rho^f \left(1 - \alpha\right)' \left(c_t^{PU}\right)\right] \frac{1}{1 + \pi_t} \tag{45}$$

$$u\left(c_t^{PU}\right) > \beta E_t \left[\left(1 - \rho^f\right) u'\left(c_t^A\right) + \rho^f \left(1 - \rho\right) u'\left(c_t^{PE}\right) + \rho^f \rho u'\left(c_t^{PU}\right)\right] \frac{1}{1 + \pi_t} \tag{46}$$

These four conditions state that the return on money is too low for NM, U, PU and PE households for them to use it to smooth consumption. It will be shown that these conditions are fulfilled in steady state. It will then be assumed that shocks are small enough such that these conditions are fulfilled in the dynamics.

4.2   Euler equations

In the equilibrium under consideration, one can use first order conditions and the envelop theorem to find the Euler equations determining the money choices of N-households in the E island. One finds

$$u'\left(c_t^{NE}\right) = \beta E_t \left[\alpha u'\left(c_{t+1}^{NE}\right) + (1 - \alpha) u'\left(c_{t+1}^{NU}\right)\right] \frac{1}{1 + \pi_{t+1}} \tag{47}$$

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One observes that the the money demand of households in the E island is made to provide resources to households moving to the U island (with probability $1 - \alpha$). One can show that if $\alpha = 1$, then the money demand of households in the E island would be 0. This money demand has the same logic as the one derived in a Bewley environment. The gain of the assumptions about the family strucure is that the distribution of money is trivial, with only two mass points in each period. This simplifies considerably the aggregation problem.

Along the same lines, participating households in the A-island have a non-trivial portfolio choice. One finds the three Euler equations:

$$u'\left(c_t^A\right) = \beta E_t \left(1 + r_{t+1}\right) u'\left(c_{t+1}^A\right) \tag{48}$$

$$u'\left(c_t^A\right) = \beta E_t \frac{1 + i_t}{1 + \pi_{t+1}} u'\left(c_{t+1}^A\right) \tag{49}$$

$$u'\left(c_t^A\right) = \beta E_t \left[\alpha^f u'\left(c_{t+1}^A\right) + \left(1 - \alpha^f\right)\left(n^{PE} u'\left(c_{t+1}^{PE}\right) + \left(1 - n^{PE}\right) u'\left(c_{t+1}^{PU}\right)\right)\right] \frac{1}{1 + \pi_{t+1}} \tag{50}$$

The first two equations are the choice of claims on the capital stock and on bonds. The third equation is the money choice of agents in the A island, which takes into account the fact that money can be used by households moving to other islands. As households in the A island cannot bring their stock or bonds to other islands, there is no self-insurance motive for these two assets. As a consequence, the Euler equations for stock and bonds are the same as the ones of a representative agent. This, again, will simplify the structure of the equilibrium.

4.3 Production side and market equilibria

The production side is similar to the one described in the simple model. For the sake of generality, it is now assumed that capital does not fully depreciate in production, the depreciation rate being λ. Profit maximization is $\max_{K,L} A_t K^\alpha L^{1-\alpha} - w_t L - (r_t + \lambda) K$. First order conditions for the firm are (8), as before, and

$$r_t + \lambda = \alpha A_t K_t^{\alpha-1} L^{1-\alpha} \tag{51}$$

with $A = e^{a_t}$, and the process for a_t given by (25). The State and monetary policies are the same as in the previous Section. Hence, the laws of motion of the money stock and taxes are given by (11) and (13).

The capital and bond market equilibria are

$$\Omega^P n^A b_t^A = b_t^o, \tag{52}$$

$$\Omega^P n^A k_t^A = K_t, \tag{53}$$

instead of (15) and (16). The two previous equations state that only $P-$households hold interest-bearing assets.

The goods market equilibrium is

$$\Omega^{NM} c_t^{NM} + (1-n)\, \Omega^{NP} c_t^{NPU} + n\Omega^{NP} c_t^{NPE} + n^{PU} \Omega^P c_t^{PU} + n^{PE} \Omega^P c_t^{PE}$$
$$+ n^A \Omega^P c_t^A + K_{t+1} = A_t K_t^\alpha L^{1-\alpha} + (1-\lambda)\, K_t + \Omega^{NM} \delta^{NM} + (1-n)\, \delta^{NP} \left(\Omega^{NP} + \kappa\Omega^P \right)$$

The labor market is

$$L = n \left(\Omega^{NP} + \Omega^P \kappa \right) \tag{54}$$

The previous equation states that labor demand is equal to labor supply in efficient units. Finally, the money market equilibrium is

$$m_t^{tot} = \Omega^N n \tilde{m}_t^{NE} + \Omega^P n^A \tilde{m}_t^P \tag{55}$$

For a given monetary policy, the program of agents and market equilibria are now fully specified. For each economy, the steady state is thus defined as equilibria for which the economy does not face technology shocks $A_t = 1$, and where the quantity of money is constant. We now present the program of the central planner.

4.4 Constrained efficient monetary policy

This Section now derives the optimal monetary policy in the economy with unemployment and participation frictions. The social welfare function has the following structure. The social planner gives a Pareto weight ω^{NM} to $NM-$households, a weight ω^N to N- households and a weight ω^P to P households.

The instrument of the central planner is the quantity of money created in each period m_t^{CB}. The Ramsey program for the central planner is the following maximization

$$W^{CE} = \max_{\{m_t^{CB}\}_{t=0..\infty}} \sum_{t=0}^{\infty} \beta^t \left[\left[\omega^{NM} \Omega^{NM} u \left(c_t^{NM} \right) + \omega^N \Omega^N \left(n^{NE} u \left(c_t^{NE} \right) + n^{NU} u \left(c_t^{NU} \right) \right) + \right] \right. \tag{56}$$

$$\left. + \omega^P \Omega^P \left(n^A u \left(c_t^A \right) + n^{PE} u \left(c_t^{PE} \right) + n^{PU} u \left(c_t^{PU} \right) \right) \right] \tag{57}$$

subject to four Euler equations (47)-(50), the five budget constraints (35), (37), (38), (39), (40) and (41), the first order conditions for the firm (8) and (51), the law of motion of the quantity of money (11), the budget of the State (13), and the five market equilibria (52)-(55), and subject to the law of motion of the technology shock given by (25).

For a given initial capital stock K_0, these equations constrain the value of $\{c_t^{NM}, c_t^{NE}, c_t^{NU}, c_t^A, c_t^{PE}, c_t^{PU}, m_t^{NPE}, m_t^A, b_{t+1}^A, k_{t+1}^A, K_{t+1}, r_t, w_t, m_t^{tot}, \tau_t\}_{t=0..\infty}$ that the planner can implement. To assess the distortion of the market economy, we now provide the first best allocation.

4.5 First Best

In the first best allocation, the central planner can provide the same consumption level to NM, N and P households. As before, we note $\tilde{x}_t$ for the value of x_t chosen by the central planner. The central planner now chooses the consumption of NM, NP and P households, $\tilde{c}_t^{NM}, \tilde{c}_t^{NP}$ and $\tilde{c}_t^P$. Its objective is thus

$$W^{FB} = \max_{\{\tilde{c}_t^{NM}, \tilde{c}_t^{NP}, \tilde{c}_t^P, \tilde{K}_{t+1}\}_{t=0..\infty}} \sum_{t=0}^{\infty} \beta^t \left[\omega^{NM}\Omega^{NM} u\left(\tilde{c}_t^{NM}\right) + \omega^N\Omega^N u\left(\tilde{c}_t^N\right) + \omega^P\Omega^P u\left(\tilde{c}_t^P\right) \right] \quad (58)$$

Subject to the budget constraint

$$\tilde{K}_{t+1} + \Omega^{NM}\tilde{c}_t^{NM} + \Omega^{NP}\tilde{c}_t^{NP} + \Omega^P\tilde{c}_t^P$$
$$= A_t\tilde{K}_t^\alpha L^{1-\alpha} + (1-\lambda)\tilde{K}_t + \Omega^{NM}\delta^{NM} + (1-n)\delta^{NP}\left(\Omega^{NP} + \kappa\Omega^P\right)$$

The following proposition presents a simple result in the general model.

Proposition 7 *If $\alpha^f < 1$, optimal monetary policy cannot restore the first best.*

The monetary policy can not restore the first best and will only be constrained effecient. monetary policy alone can not cancel the effect of incomplete markets.

4.6 Calibration

We now provide a calibration to compare three different economies. The first economy is the economy where the central planner set a constrained efficient monetary policy rule by solving program (56). The second economy is the economy where the central planner is unconstrained and can implement the first best allocation. The third economy is an economy where monetary policy is inactive. In this economy, we impose that m_t^{CB} is 0. For this comparison to be meaningful, I choose Pareto weights such that the optimal inflation rate in the constrained efficient steady-state allocation is 0. As before, this strategy implies that the welfare gain of an

active monetary policy is only the result of its ability to affect the business cycle and is not the outcome of a reduction in steady-state distortions.

The period is a quarter. The share of NM households is set to $\Omega^{NM} = 10\%$, the share of N households is 40% and the share of P households is set to 50%. These shares are motivated by the previous discussion of the data.

Preference parameters are set to standard values. The discount factor is $\beta = 0.99$ and the curvature of the utility function is $\sigma = 1.5$. The production function is such that the capital share is $\mu = .36$ and the depreciation rate is $\lambda = 0.025$. The discount factor determines the steady-state interest rate $1 + r = 1/\beta$, with equation (48). This and the depreciation rate determine the steady-state capital stock and the steady-state wage rate w per efficient unit.

Concerning the labor market, a quarterly job separation rate and job finding rate is estimated using Shimer (2005) methodology, as in Challe and Ragot (2014). The quarterly job separation rate is 5%, such that $\alpha = 0.95$, and the quarterly job finding rate is 79%, such that $\rho = 0.21$. The replacement rate is calibrated to match a fall in the consumption of households falling into unemployment equal to 10%. This value is an intermediate value between the ones found by Cochrane (1991) and by Gruber (1997). This implies a replacement rate of 0.85, which is an intermediate value between the ones used by Shiller (2005) and Hagerdorn and Manovski (2009). As a consequence, I consider $\delta^N/w = 0.85$ in steady state.

Concerning inequality in income, I use the 2004 Survey of Consumer Finance to match income inequality between the three groups[10]. The ratio of the income of the top 50% over the income of the 10-50% is 4.42. The ratio of income of the 10-50% over the income of the bottom 10% is 3.66. These ratios will be used to calibrate the model. I take $\kappa = 4.42$, and $\delta^{NM} = w/3.66$.

Two parameters, α^f and ρ^f, concern the participation structure in financial markets. To my knowledge there is no direct estimation of the participation frequency of households in financial markets. I follow the strategy of Alvarez et al. (2009) which is to calibrate participation frequency to match some monetary moments of the data. First, as an agnostic calibration, I set $\alpha^f = \rho^f$. Second, I use again the SCF 2004 to measure the quantity of money (M1) held by households as a fraction of their annual disposable income. This fraction is 8% (Ragot,

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[10]The 2004 SCF survey is used to avoid the high house prices of the 2007 survey and the low nominal interest rate in the 2010 survey. Nevertheless, it has been checked that the shape of the distribution does not vary a lot between the various surveys.

2014). It implies a value $\alpha^f = 0.85$. This calibration strategy implies that half of the population of $P-$households participates in financial markets in each period. The probability not to participate next period, when participating is 15%.

The process for technology is set to standard values. The persistence of technology shock is set to $\rho^a = 0.95$ and the standard deviation is $\sigma_a = 1\%$.

The last parameters to be determined are the Pareto weights ω^{NM}, ω^N and ω^P. First, as a normalization, I set $\omega^P = 1$. Second, I impose as a benchmark $\omega^{NM} = \omega^N$, and I choose ω^{NM} such that the optimal inflation in the steady-state Ramsey problem is 0. Solving numerically the model, one finds $\omega^{NM} = \omega^N = 4.37\%$. Next Table summarizes parameter values.

Population (%) and Pareto weight					Preferences and technology					
Ω^{NM}	Ω^N	Ω^P	ω^P	$\omega^{NM}(\%)$	β	σ	μ	λ	ρ^a	σ_a
10	40	50	1	4.37	.99	1.5	.36	.025	.95	.01
Income structure					Uninsurable risk					
κ	δ^N/w	δ^{NM}/w			α	ρ	α^f	ρ^f		
4.42	0.85	1/3.66			.95	.22	.85	.85		

Table 1: Parameter values. See text for description.

The next table summarizes the steady state outcome of the model. First, the model reproduces standard aggregate quantity and prices. The ratio of money over households disposable income is 8% as in the data. The consumption and money levels are provided and the Gini coefficient for consumption and money are computed. This simple model does a good job in reproducing consumption and money inequalities. The Gini coefficient for consumption (Gini C.) is 0.37, very close to its empirical counterpart which is 0.34. The Gini coefficient for money (Gini M) is 0.73, which is again close to its empirical value, which is 0.8 (see Ragot 2014 for a discussion)[11].

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[11]It is checked that the four conditions (43)-(46) are fulfilled.

Aggregate quantities, prices and inequalities							
K	n	L	r	w	M/Y	Gini C.	Gini M.
93.5	.94	2.36	1%	2.37	8%	0.37	0.73
Consumption and money levels							
c^{NM}	c^{NE}	c^{NU}	c^A	c^{PE}	c^{PU}	m^{NE}	m^P
0.64	2.26	2.08	12.5	12.1	10.0	0.23	9.92

Table 2: Model outcome

To understand the trade-offs faced by monetary policy in this environment, Fig. 2 plots the reaction of threeconomies to the same technology shock.
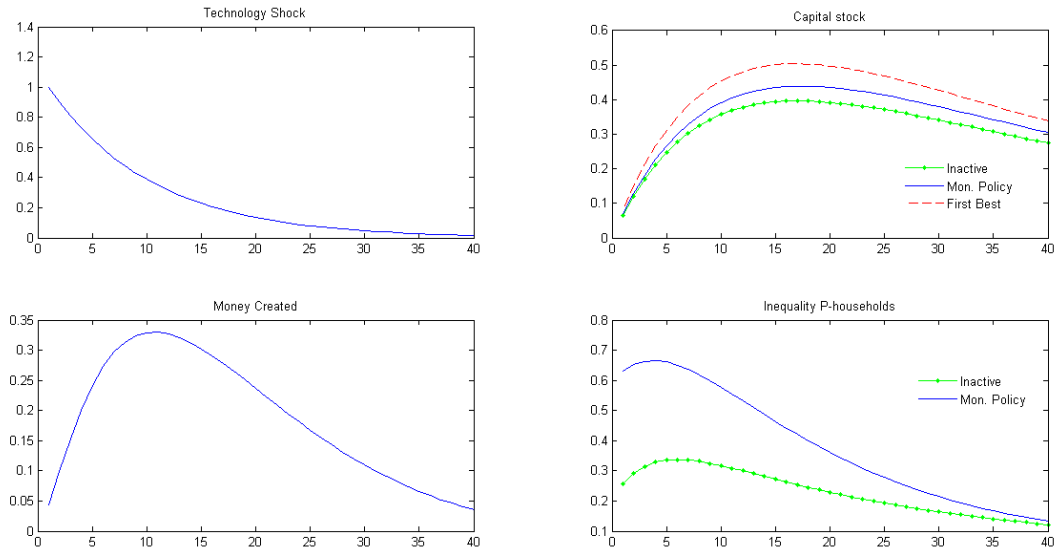

Fig 2: Outcomes of the three economies after the same technology shock. The green dotted line is the market economy with inactive monetary policy ($m^{CB} = 0$), the blue solid line is the market economy with a constrained efficient monetary policy. The red dotted line is the first best allocation.

The first panel presents the technology shock (A_t), as percentage deviation to the steady state value. The second panel presents the capital stock in the three economies. The first economy is the one with an inactive monetary policy ($m^{CB} = 0$). It is represented by a green dotted line. The second economy is the one where monetary policy is optimally designed. It is plotted with a blue solid line. The third economy is the first best allocation. It is represented

by the red dashed line. When monetary policy is inactive, one can observe that the capital stock increases much less than in the first-best allocation. The reason for this lack of capital accumulation has been explained in the previous Section, notably in Proposition 2.

Optimal monetary policy contributes to increase the capital stock after a positive technology shock. In the economy with active monetary policy ,the capital stock is 9% higher at the peak compared to the economy with inactive monetary policy. This capital stock remains 10% smaller than the one in the first-best allocation. The third panel presents the path of the quantity of money created by the monetary authority to achieve this allocation. Money created is hump-shaped. One can notice that optimal monetary policy can not implement the first best allocation. To understand the trade-offs faced by monetary authorities, the fourth panel plots the change in the consumption inequality between households in the A island and households in the PU island (i.e. c_t^A/c_t^{PU}). One observes that the increase in consumption inequality is much higher when monetary policy is active than when it is not. The change in consumption inequality for the first-best allocation is not plotted because it is 0. To induce an increase in capital accumulation, optimal monetary policy transfers some wealth to participating households. This increases their savings but it increases consumption inequality (what decreases Social Welfare). Optimal monetary policy is the result of the trade-offs between these two effects.

To summarize the quantitative results of the model, optimal monetary policy increases capital accumulation by 10% in the business cycles, at the cost of higher consumption inequality. This result is different from the ones found in heterogeneous agents literature, when capital accumulation is not introduced (Kehoe, Levine and Woodford, 1990; Lippi, Ragni and Trachter 2013 among others). In this last literature, the role of monetary policy is to decrease consumption inequality and thus to provide insurance to households. The introduction of capital accumulation changes the results, as the main objective of monetary policy is to affect the incentives to save.

5   Conclusion

Households facing limited participation in financial markets smooth consumption using money. As the return on money is different from the marginal return to capital, accumulationis not optimal. Active monetary policy can improve the incentives to save both in the business cycle and at the zero lower bound. The trade-off faced by monetary policy is between restoring proper

incentives to save and increasing consumption inequality. Optimal monetary policy is shown to increase capital accumulation by 10% after a technology shock.

The analysis has been performed with flexible prices, to be able to provide forma proof. The next step is to introduce nominal frictions in the environment considered in this paper. Such frictions generate additional distortions in the consumption-saving choices of households. The understanding of the interactions between these distortions may be key to provide a deeper understanding of optimal monetary policy in the business cycle.

6 References

Algan, Y., Ragot, X. (2010)., Monetary Policy with Heterogeneous Agents and Borrowing Constraints. Review of Economic Dynamics, 13, 295-316.

Alvarez Fernando, Andrew Atkeson & Chris Edmond, (2009). "Sluggish Responses of Prices and Inflation to Monetary Shocks in an Inventory Model of Money Demand," The Quarterly Journal of Economics, MIT Press, vol. 124(3), pages 911-967, August.

Alvarez, Fernando, Andrew Atkeson & Patrick J. Kehoe, (2002). "Money, Interest Rates, and Exchange Rates with Endogenously Segmented Markets," Journal of Political Economy, University of Chicago Press, vol. 110(1), pages 73-112, February.

Alvarez, Fernando & Atkeson, Andrew, (1997). "Money and exchange rates in the Grossman-Weiss-Rotemberg model," Journal of Monetary Economics, Elsevier, vol. 40(3), pages 619-640, December.

Alvarez, Fernando E. and Francesco Lippi. (2009). "Financial Innovation and the Transactions Demand for Cash." Econometrica 77 (2):363–402.

Alvarez, Fernando, and Lippi Francesco (2013), "The demand of liquid assets with uncertain lumpy expenditures", Journal of Monetary Economics, 2014, forthcoming.

Alvarez, Fernando and Francesco Lippi, (2014). "Persistent Liquidity Effects and Long-Run Money Demand," American Economic Journal: Macroeconomics, American Economic Association, vol. 6(2), pages 71-107, April.

Cao, Shutao, Cesaire A. Meh, Jose-Vıctor Rıos-Rull and Yaz Terajima (2012), "Inflation, Demand for Liquidity and Welfare", Working Paper.

Chamley, C. (2012), "A paradox of thrift in general equilibrium without forward markets", Journal of the European Economic Association, 10, 1215–1235.

Chamley, C. (2013), "Demand creates its own supply", Review of Economic Studies, fort-coming.

Christiano, Lawrence J., and Martin Eichenbaum. (1995). Liquidity effects, monetary policy and the business cycle, Journal of Money, Credit and Banking. 27(4, part 1): 1113-1136.

Cooley, Thomas and Gary D. Hansen (1989), "Inflation Tax in a Real Business Cycle Model", The American Economic Review, Vol. 79, No. 4. (Sep., 1989), pp. 733-74

Erosa, A., Ventura,G., (2002) "On Inflation as a Regressive Consumption Tax". Journal of Monetary Economics 49, 761-795.

Fuerst, Timothy (1992), "Liquidity, loanable funds and economic activity", Journal of Monetary Economy, 29(1),3-24.

Grossman S.J and Weiss L. (1983), A transactions-based model of the monetary transmission mechanism, American . Econic. Review. 73 (1983). 871-880.

Hayek, F. (1931), "Price and Production", Augustus M Kelley Pubs; 2 Revised edition (June 1967)

Heathcote, Jonathan, Storesletten, Kjetil and Violante, Giovanni (2014). "Consumption and Labor Supply with Partial Insurance: An Analytical Framework," American Economic Review, American Economic Association, vol. 104(7), pages 2075-2126, July.

Heller, W., (1974). The Holding of Money Balances in General Equilibrium. Journal of Economic Theory 7, 93-108.

Kahn, A. Thomas J. (2011), "In‡ation and Interest Rates with Endogenous Market Segmentation", working paper.

Kehoe, T David K. Levine and Michael Woodford, (1990). "The optimum quantity of money revisited," Working Papers 404, Federal Reserve Bank of Minneapolis.

Keynes, J. M. (1936) " The General Theory of Employment, Interest and Money by John Maynard Keynes" ,Macmillan Cambridge University Press

Krusell, P. and Smith, A.A. (1998). 'Income and wealth heterogeneity in the macroeconomy', Journal of Political Economy, vol. 106(5), pp. 867-896.

Lippi, Farncesco, Ragni Stefania and Trachter, Nicholas, (2013), "State dependant monetary policy", Working Paper.

Lucas Robert, (1990), "Liquidity and Interest Rates", Journal of Economic Theory, 50, 237 - 264.

Paiella, M., (2001). Limited Financial Market Participation: A transaction cost-based Explanation. IFS working paper 01/06.

Ragot, X. (2014) "The case for a financial approach to money demand", Journal of Monetary Economics, in press.

Rotemberg J.(1984) A monetary equilibrium model with transactions costs. Journal of Political Economy.92 (1984). 458.

Telyukova,I., Wright,R., (2008). A Model of Money and Credit, with Application to the Credit Card Debt Puzzle. Review of Economic Studies 75, 629-647.

Vissing-Jorgensen, A., (2002). Towards an explanation of the households portfolio choice heterogeneity: Non financial income and participation cost structure, NBER Working Paper 8884.

Vissing-Jorgensen, A., (2002), "Limited Asset Market Participation and the Elasticity of Intertemporal Substitution", Journal of Political Economy.

Williamson S. (2008). "Monetary policy and distribution", Journal of Monetary Economics 55 1038–1053.

A   Proof of Proposition 1

Part 1). In steady state $(A_t = 1)$, the Euler equation of P-households (5) pins down the real interest rate $1 + r^* = \frac{1}{\beta}$. Using the first order condition for the firm (9) One finds $K^* = L(\Omega)(\beta\mu)^{1/(1-\mu)}$. As a consequence, $K^* = \tilde{K}^*$ and $Y = \tilde{Y}^*$. Using the same equations for the optimal program (Equations **??** and **??**), one finds the same value. Using equations (1)-(13), one finds the consumption of N-households, which depends on the steady state inflation rate.

$$c^{n*} = (1 - \mu)(\beta\mu)^{\frac{\mu}{1-\mu}}\left(1 - \frac{\pi(\beta - \Omega)}{(1+\pi)\beta - \frac{\Omega}{2}}\right),$$

The consumption of P-households is given by the goods market equilibrium (17).

Part 2). the central planner allocation implies

$$\frac{\tilde{c}^{p*}}{\tilde{c}^{n*}} = \omega_p^{-\frac{1}{\sigma}}$$

The market allocation is, when $\pi = 0$:

$$c^{p*} = \left(\mu(1-\beta)\frac{\Omega/2 + 1 - \Omega}{1 - \Omega} + 1 - \mu\right)(\mu\beta)^{\frac{\mu}{1-\mu}} \text{ and } c^{n*} = (1 - \mu)(\mu\beta)^{\frac{\mu}{1-\mu}}$$

As total consumption is the same in the market economy and for the optimal allocation (because output and capital are the same) , a necessary and sufficient condition to have $c^{n*} = \tilde{c}^{n*}$ and $c^{p*} = \tilde{c}^{p*}$ is $\tilde{c}^{p*}/\tilde{c}^{n*} = c^{p*}/c^{n*}$. Using the three previous equations, this condition can be written as

$$\omega_p = \left(1 + \frac{\mu}{1-\mu}(1-\beta)\frac{\Omega/2 + 1 - \Omega}{1 - \Omega}\right)^{-\sigma}$$

B   Intertemporal budget constraints

Using the capital market (16) and the first order condition (9), the budget constraint (**??**) can be written as

$$k_{t+1}^p + c_t^p = \Theta(\Omega)(1 + r_t)k_t^p$$

where $\Theta(\Omega) = \frac{1-\mu}{\mu}\frac{1}{1-\frac{\Omega}{2}} + \frac{1}{1-\Omega}$. Iterating forward this constraint, the equality (16) and using the transversality condition one finds expression (26).

Define as $1 + \tilde{r}_t \equiv \mu A_t \tilde{K}_t^{\mu-1} L(\Omega)^{1-\mu}$ as the marginal productivity of capital for the optimal allocation. The budget constraint of the Central Planner (**??**) can be written as

$$\left(\frac{\Omega}{2}\omega_p^{-\frac{1}{\sigma}} + 1 - \Omega\right)\tilde{c}_t^p + K_{t+1} = \frac{1 + \tilde{r}_t}{\mu}K_t$$

Iterating forward one finds the expression (27), with $\Theta(0) = 1/\mu$ and with $\Psi(\Omega) = 1 + \frac{1}{2}\frac{\Omega}{1-\Omega}(\omega_p)^{-\frac{1}{\sigma}}$.

C   Proof of Propositions 2

Under the maintained assumption that $m_t^{CB} = \tau_t = b_t^g = 0$, the model is simply

$$u'(c^p) = E_t(1 + r_{t+1})u'(c_{t+1}^p)$$

$$k_{t+1}^p + c_t^p = w_t + (1 + r_t)k_t^p$$

$$c_t^n = \frac{m_{t-1}^n}{1 + \pi_t} = m_t^n = w_t$$

With the first order conditions (9), (8) and the capital market equilibrium (16). Linearization of the model around the steady state gives

$$E_t \hat{c}^p_{t+1} - \hat{c}^p_t = \frac{\mu - 1}{\sigma} \hat{K}_{t+1} + \frac{1}{\sigma} E_t a_{t+1} \tag{59}$$

$$\hat{K}_{t+1} + (\Psi(\Omega) - 1) \hat{c}^p_t = \theta(\Omega) \left(\mu \hat{K}_t + a_t \right) \tag{60}$$

$$\hat{c}^n_t = a_t + \mu \hat{K}_t$$

where the function $\theta(\Omega)$ is defined as

$$\theta(\Omega) \equiv \frac{1}{\beta} \left(1 + (1 - \Omega) \frac{1 - \mu}{\mu L(\Omega)} \right) > 1$$

Substituting $\hat{c}^p_t$ in (59) using 60), one finds one equation in the variable $\hat{K}_t$. Using the method of unknown coefficient, one finds that the capital stock as the form

$$\hat{K}_{t+1} = B(\sigma, \theta(\Omega)) \hat{K}_t + D^a(\sigma, \theta(\Omega), \rho^a) a_t$$

where ρ^a is the persistence of the technology shock, and where

$$B(\sigma, \theta) = \frac{1}{2\sigma} ((1 - \mu(1 - \sigma)) \theta + \mu + \sigma - 1)$$
$$- \frac{1}{2} \sqrt{\frac{1}{\sigma^2} ((1 - \mu(1 - \sigma)) \theta + \mu + \sigma - 1)^2 - 4\theta\mu}$$

$$D^a(\sigma, \theta, \rho) = \frac{\theta + \frac{\rho}{\sigma} (\theta(1 - \sigma) - 1)}{\frac{1}{\sigma} ((1 - \mu(1 - \sigma)) \theta + \mu + \sigma - 1) - B(\sigma, \theta) - \rho}$$

The linearization of the Central Planer program yields

$$E_t \widetilde{\hat{c}}^p_{t+1} - \widetilde{\hat{c}}^p_t = \frac{\mu - 1}{\sigma} \widehat{\tilde{K}}_t + \frac{1}{\sigma} E a_{t+1} \tag{61}$$

$$\left(\frac{1}{\mu\beta} - 1 \right) \widetilde{\hat{c}}^p_t + \widehat{\tilde{K}}_{t+1} = \frac{1}{\mu\beta} \left(a_t + \mu \widehat{\tilde{K}}_{t+1} \right) \tag{62}$$

First comparing (59)-(60) and (61)-(62), note that the optimal and the market allocation are the same when $\Omega = 0$ because $\theta(0) = 1/(\mu\beta)$. One finds thus directly find the optimal low of motion of the capital stock is

$$\widehat{\tilde{K}}_{t+1} = \tilde{B} \widehat{\tilde{K}}_t + \tilde{D}^a a_t \text{ with } \tilde{B}, \tilde{D}^a > 0$$

with

$$\tilde{B} = B \left(\sigma, \frac{1}{\mu\beta} \right) \text{ and } \tilde{D}^a = D^a \left(\sigma, \frac{1}{\mu\beta}, \rho \right)$$

Note that when $\sigma = 1$, then

$$B(1, \theta) = \mu \text{ and } D(1, \theta, \rho) = 1$$

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As the consequence, the dynamics of the capital stock is the same in both economy. It is then easy to show that the consumption of both P and N-households is the same in both economies, what conclude the first part of the Proposition. For the second part, Assume that $\sigma = 1 + \varepsilon$ with ε small such that a first order expansion in ε relevant. One finds

$$B\left(\varepsilon, \theta\right) = \mu + \left(\mu - 1\right)\left(\theta - 1\right)\frac{1}{2}\left(1 - \frac{1}{\left(\theta + \mu\right)\left(\theta - \mu\right)}\right)\varepsilon$$

$$D^a\left(\varepsilon, \theta, \rho\right) = 1 + \frac{\theta - 1}{\theta - \rho}\left(\left(1 - \mu\right)\frac{\theta}{\theta - \mu} - \rho\right)\varepsilon$$

With the notations of the Proposition

$$\frac{\partial \hat{K}}{\partial \varepsilon^a} = D^a\left(\varepsilon, \theta\left(\Omega\right), \rho\right) \text{ and } \frac{\partial \widehat{\tilde{K}}}{\partial \varepsilon^a} = D^a\left(\varepsilon, \frac{1}{\mu\beta}, \rho\right)$$

Using the expression of $D^a\left(\varepsilon, \theta, \rho\right)$ one finds for $\rho = 1$, $D^a\left(\varepsilon, \frac{1}{\mu\beta}, 1\right) < D^a\left(\varepsilon, \theta\left(\Omega\right), 1\right)$ and for $\rho = 0$, $D^a\left(\varepsilon, \frac{1}{\mu\beta}, 0\right) > D^a\left(\varepsilon, \theta\left(\Omega\right), 0\right)$. As the function $D^a\left(\varepsilon, \theta\left(\Omega\right), \rho\right)$ is continuous for $\rho < 1 < \theta$, one finds the result of the Proposition.

D Proof of Proposition 3

Define as $\tilde{g}\left(A_t, \tilde{K}_t\right)$ the optimal decision rule of the central planner : $\tilde{K}_{t+1} = \tilde{g}\left(A_t, \tilde{K}_t\right)$, solving the program (**??**) -(**??**).

Assume that the money supplied follows the rule

$$
\begin{aligned}
m_t^{CB} = & \left(\left(\left(\left(1 - \mu\right)\frac{1 - \Omega}{1 - \frac{\Omega}{2}} + \mu\right)\left(K^*\right)^\mu L^{1-\mu} - K^*\right)\left(\frac{A_t\tilde{K}_t^\mu L^{1-\mu} - \tilde{g}\left(A_t, \tilde{K}_t\right)}{\left(K^*\right)^\mu L^{1-\mu} - K^*}\right)\right. \\
& - \left(\left(1 - \mu\right)\frac{1 - \Omega}{1 - \frac{\Omega}{2}} + \mu\right)A_t K_t^\mu L^{1-\mu} + \tilde{g}\left(A_t, K_t\right) + \Omega\left(1 + i_{t-1}\right)m_{t-1}^{CB}\frac{\left(1 - \mu\right)A_t K_t^\mu L^\mu}{m_{t-1}^n - \left(1 + i_{t-1}\right)m_{t-1}^{CB}}\right) \\
& * \left(1 - 2\frac{\left(1 + i_{t-1}\right)m_{t-1}^{CB}}{m_{t-1}^n - \left(1 + i_{t-1}\right)m_{t-1}^{CB}}\right)^{-1}
\end{aligned}
\tag{63}
$$

Although this expression is complex, it is only a function of the past state variable $K_t, m_{t-1}^{CB}, m_{t-1}^n, i_{t-1}$ and on the current technology shock A_t. It is shown that the first best allocation is a solution of the program of all agents. As a consequence, optimal monetary policy can implement the first best. The basic idea of the proof is to design a monetary policy such that the budget constraint

of participating households in the limited-participation economy is the same as the one in the optimal economy. The proof is done in three steps.

First, using the equations (1), (11), (14), (13), one finds that

$$\Omega \frac{1 + i_{t-1}}{1 + \pi_t} m_{t-1}^{CB} = \Omega \left(1 + i_{t-1}\right) m_{t-1}^{CB} \frac{\left(1 - \mu\right) A_t K_t^\mu L^\mu}{m_{t-1}^n - \left(1 + i_{t-1}\right) m_{t-1}^{CB}} + 2 \frac{\left(1 + i_{t-1}\right) m_{t-1}^{CB}}{m_{t-1}^n - \left(1 + i_{t-1}\right) m_{t-1}^{CB}} m_t^{BC}$$

Using the previous equation to substitute for the denominator in (63), the optimal policy rule can be written as

$$m_t^{CB} = H\left(\Omega, A_t, K_t\right) + \Omega \frac{1 + i_{t-1}}{1 + \pi_t} m_{t-1}^{CB}$$

where,

$$H\left(\Omega, A_t, K_t\right) \equiv \left(\left(\left(1 - \mu\right) \frac{1 - \Omega}{1 - \frac{\Omega}{2}} + \mu\right) \left(K^*\right)^\mu L^{1-\mu} - K^*\right) \left(\frac{A_t \tilde{K}_t^\mu L^{1-\mu} - \tilde{g}\left(A_t, \tilde{K}_t\right)}{\left(K^*\right)^\mu L^{1-\mu} - K^*}\right)$$
$$- \left(\left(1 - \mu\right) \frac{1 - \Omega}{1 - \frac{\Omega}{2}} + \mu\right) A_t K_t^\mu L^{1-\mu} + \tilde{g}\left(A_t, K_t\right)$$

Second, using the budget constraint of participating households (3) and (13), one finds that the program of participating households can be written as a simple system in c_t^p and K_t.

$$u'\left(c^p\right) = \beta E_t \left(\mu A_{t+1} K_{t+1}^{\mu-1} L^{1-\mu}\right) u'\left(c_{t+1}^p\right)$$
$$\left(1 - \Omega\right) c_t^p = \left(\left(1 - \mu\right) \frac{1 - \Omega}{1 - \frac{\Omega}{2}} + \mu\right) A_t K_t^\mu L^{1-\mu} - \Omega \frac{1 + i_{t-1}}{1 + \pi_t} m_{t-1}^{CB} - K_{t+1} + m_t^{CB}$$

Plugging the expression of the function H in the budget constraint, one finds

$$\left(1 - \Omega\right) c_t^p = \frac{\left(\left(\left(1 - \mu\right) \frac{1 - \Omega}{1 - \frac{\Omega}{2}} + \mu\right) \left(K^*\right)^\mu L^{1-\mu} - K^*\right)}{\left(K^*\right)^\mu L^{1-\mu} - K^*} \left(A_t K_t^\mu L^{1-\mu} - \tilde{g}\left(A_t, K_t\right)\right) + \tilde{g}\left(A_t, K_t\right) - K_{t+1}$$

As the the consumption of participating households is optimal in steady state, we have

$$\tilde{c}^{p*} = \frac{\left(K^*\right)^\mu L^{1-\mu} - K^*}{\frac{\Omega}{2} \left(\omega_p\right)^{-\frac{1}{\sigma}} + \left(1 - \Omega\right)} = c^{p*} = \frac{\left(\left(1 - \mu\right) \frac{1 - \Omega}{1 - \frac{\Omega}{2}} + \mu\right) \left(K^*\right)^\mu L^{1-\mu} - K^*}{1 - \Omega}$$

Using the two previous equations yield

$$\left(\frac{\Omega}{2} \left(\omega_p\right)^{-\frac{1}{\sigma}} + \left(1 - \Omega\right)\right) c_t^p = A_t K_t^\mu L^{1-\mu} - \tilde{g}\left(A_t, K_t\right) + \frac{\frac{\Omega}{2} \left(\omega_p\right)^{-\frac{1}{\sigma}} + \left(1 - \Omega\right)}{1 - \Omega} \left(\tilde{g}\left(A_t, K_t\right) - K_{t+1}\right)$$

Third, wen can show that the optimal decision rule $K_{t+1} = \tilde{g}(A_t, K_t)$ is a solution to the problem of participating households. The program of these households can be written as

$$u'(c_t^p) = \beta E_t \left(\mu A_{t+1} K_{t+1}^{\mu-1} L^{1-\mu} \right) u'(c_{t+1}^p)$$

$$K_{t+1} + \left(\frac{\Omega}{2} (\omega_p)^{-\frac{1}{\sigma}} + (1-\Omega) \right) c_t^p = A_t K_t^\mu L^{1-\mu} + \frac{1}{2} \frac{\Omega}{1-\Omega} (\omega_p)^{-\frac{1}{\sigma}} (\tilde{g}(A_t, K_t) - K_{t+1})$$

One recognizes the program of the Central planer (??)-(??), with an extra term at the right hand side $\frac{1}{2} \frac{\Omega}{1-\Omega} (\omega_p)^{-\frac{1}{\sigma}} (\tilde{g}(A_t, K_t) - K_{t+1})$, which is nul when $K_{t+1} = \tilde{g}(A_t, K_t)$. As a consequence, if $K_{t+1} = \tilde{g}(A_t, K_t)$ is a solution of the Central Planner program, it is also a solution of the program of P agents in the limited participation economy. Hence, $c_t^p = \tilde{c}_t^p$ and $K_t = \tilde{K}_t$ and $c_t^n = \tilde{c}_t^n$ by the goods market equilibrium.

E   Proof of proposition 4

We proceed by construction following a guess and verify strategy. Assume that the first best allocation can be implemented $\{\tilde{K}_t, \tilde{c}_t^n, \tilde{c}_t^p\}_{t=0..\infty}$.

This allocation satisfied the Euler equation of $P-$agents. Note that the two first order conditions for P agents (for capital and for bonds) imply $\frac{1+i_{t-1}}{1+\pi_t} = 1 + r_t = \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu}$ (by assumption)

As a consequence, for this allocation to satisfy the budget constraint of $P-$agents (28), we must have

$$m_0^{CB} = \Delta_0 \tag{64}$$

$$m_t^{CB} - \Omega \left(\mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu} \right) m_{t-1}^{CB} = \Delta_t, \text{ for } t \geq 1 \tag{65}$$

where

$$\Delta_t \equiv \tilde{K}_{t+1} + (1-\Omega) \tilde{c}_t^p - \tilde{Y}_t + \frac{\Omega}{2} \tilde{w}_t \tag{66}$$

(with $\tilde{Y}_t = A_t \tilde{K}_t^\mu L^{1-\mu}$ and $w_t = (1-\mu) \tilde{K}_t^\mu L^{-\mu}$, as before)

The new variable Δ_t is precisely the missing saving of $N-$agents when the optimal allocation is implemented. The sequence $\{\Delta_t\}_{t=0..\infty}$ does not depend on the market allocation but it is a simple function of the optimal one. The two constraints (64) and (65) state that monetary policy implement the transfers which exactly correspond to this missing saving. One can observe that these two constaints define uniquely recursively the monetary policy m_t^{CB} as function of the optimal allocation only. The sequences $\{\tilde{K}_t, , \tilde{c}_t^p\}_{t=0..\infty}$. solve the program of the P-agents.

The consumption of N-agents is $\{\tilde{c}_t^n\}_{t=0..\infty}$ due to the goods market equilibrium. The other equations define the remaining variables. As we assume that the ZLB doe not bind, the optimal allocation is a market allocation. Q.E.D.

F   Proof of proposition 5

As in the proof of the previous Proposition, we proceed by construction, following a guess and verify strategy.

Step 1. *Construction of the money supply.* Assume that monetary policy is able to implement the missing transfer defined in (66), we have from the budget constraints (28), for $t \geq 2$ and from (31) and (32), for $t = 0$ and $t = 1$.

$$m_0^{CB} - (1 - \Omega) m_0^p = \Delta_0 \tag{67}$$

$$m_1^{CB} - \Omega \frac{1}{1 + \pi_1} m_0^{CB} + (1 - \Omega) \frac{m_0^p}{1 + \pi_1} = \Delta_1 \tag{68}$$

$$m_t^{CB} - \Omega \frac{1 + i_{t-1}}{1 + \pi_t} m_{t-1}^{CB} = \Delta_t, \text{ for } t = 2..\infty \tag{69}$$

For the first best allocation to be a market allocation we must have

$$1 + \pi_1 = (1 + r_1)^{-1} = \left(\mu A_1 \tilde{K}_1^{\mu-1} L^{1-\mu} \right)^{-1} \tag{70}$$

and the money market clears in all periods.

We now exhibit the condition for the existence (and uniqueness) of such a monetary policy. The money market equilibriums are

$$(1 - \Omega) m_0^p + \frac{\Omega}{2} m_0^n = \frac{\frac{\Omega}{2} m^{n*}}{1 + \pi_0} + m_0^{CB} \tag{71}$$

$$\frac{\Omega}{2} m_1^n = \frac{(1 - \Omega) m_0^p + \frac{\Omega}{2} m_0^n}{1 + \pi_1} + m_1^{CB} \tag{72}$$

$$\frac{\Omega}{2} m_t^n = \frac{\frac{\Omega}{2} m_{t-1}^n}{1 + \pi_t} + m_t^{CB}, \text{ for } t \geq 2 \tag{73}$$

Moreover, we have the money demand for N-agents is

$$m_0^n = (1 - \mu) A_0 \tilde{K}_0^\mu L^{-\mu}$$

$$m_t^n = (1 - \mu) A_t \tilde{K}_t^\mu L^{-\mu} + \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu} m_{t-1}^{CB}$$

where we have substituted the real wage by its expression as a function of the optimal capital stock and we used the fact $\frac{1+i_{t-1}}{1+\pi_t} = 1 + r_t = \mu A_t \tilde{K}_t^{\mu-1} L^{1-\mu}$ for all $t \geq 1$.

As a consequence, the period 1 money market equilibrium implies

$$\frac{\Omega}{2} \left((1-\mu) A_1 \tilde{K}_1^{\mu} L^{-\mu} + \mu A_1 \tilde{K}_1^{\mu-1} L^{1-\mu} m_0^{CB} \right) = \frac{(1-\Omega) m_0^p + \frac{\Omega}{2} (1-\mu) A_0 \tilde{K}_0^{\mu} L^{-\mu}}{1+\pi_1} + m_1^{CB} \quad (74)$$

Given the value of π_1 given by equality (70), the three equations (67), (68) and (81) are 3 linear equations in the 3 variables m_0^{CB}, m_1^{CB} and m_0^p. Solving, we find

$$m_0^p = \frac{1}{1-\Omega} \left(\left(\left(\frac{1-\mu}{\mu} \frac{\tilde{K}_1}{L} - \frac{2}{\Omega} \frac{\Delta_1}{\mu A_1} \left(\frac{\tilde{K}_1}{L} \right)^{1-\mu} \right) - \left((1-\mu) A_0 \tilde{K}_0^{\mu} L^{-\mu} + \Delta_0 \right) \right)$$

where

$$\Delta_t \equiv \tilde{K}_{t+1} + (1-\Omega) \tilde{c}_t^p - A_t \tilde{K}_t^{\mu} L^{1-\mu} + \frac{\Omega}{2} (1-\mu) \tilde{K}_t^{\mu} L^{-\mu} \quad (75)$$

For this value to be consistent with the equilibrium definition, we must have $m_0^p \geq 0$. This condition is the one stated in Proposition 5

With the value m_0^p and the sequence of equations (67)-(69), one can construct recursively, the sequence $\{m_t^{CB}\}_{t=0..\infty}$. With equations (71) and (73), one can deduce the inflation rate π_0 and π_t for $t \geq 2$

Step 2. Proof that the constructed money supply implements the first best allocation.

Under the assumption tht $m_0^p \geq 0$, one can show that the sequence $\{m_t^{CB}\}_{t=0..\infty}$.constructed in step 1, implement the first best allocation $\{\tilde{K}_t, \tilde{c}_t^n, \tilde{c}_t^p\}_{t=0..\infty}$. By construction, the budget constraint of all agents hold with equality. The goods, capital and money market equilibriums hold. The Euler equations of $N-$agents hold in all periods.

G   Proof of Proposition 6

Following the same steps as the case of open market operations, the model can be written as

$$u'\left(c_t^p\right) = E_t\left(\mu A_{t+1}K_{t+1}^{\mu-1}L^{1-\mu}\right)u'\left(c_{t+1}^p\right)$$

$$u'\left(c_t^p\right) = (1+i_t)E_t\frac{u'\left(c_{t+1}^p\right)}{1+\pi_{t+1}}$$

$$u'\left(c_t^p\right) \geq E_t\frac{u'\left(c_{t+1}^p\right)}{1+\pi_{t+1}} \text{ and } m_t^p = 0 \text{ if strict inequality}$$

$$K_{t+1} + (1-\Omega)c_t^p = \left((1-\mu)\frac{1-\Omega}{1-\frac{\Omega}{2}}+\mu\right)A_tK_t^\mu L^{1-\mu} + (1-\Omega)\left(m_t^{CB} - m_t^p + \frac{m_{t-1}^p}{1+\pi_t}\right)$$

$$(1-\Omega)c_t^p + \frac{\Omega c_t^n}{2} + K_{t+1} = A_tK_t^\mu L^{1-\mu}2$$

$$m_t^n = \frac{m_{t-1}^n}{1+\pi_t} + \frac{2}{\Omega}m_t^{BC}$$

$$m_t^n = (1-\mu)A_tK_t^\mu L^{-\mu} + m_t^{CB}$$

In this economy, the new money is a direct transfer to all agents.Assume that the economy is hit in period 0, by an unexpected shock which implies the sequence $\{A_t\}_{t=0..\infty}$ for technolgy.

The case without a binding ZLB. Define Δ_t as in (66). It is the missing saving of $NP-$households. One can easily show that the first best allocation is implemented when $m_t = \Delta_t$. One can observe how simple is this economy.

The case with a binding ZLB. When the ZLB is binding for one period, we can try to construct the optimal money supply as in the proof of Proposition 5. The equations (67)-(69) are now

$$m_0^{CB} - m_0^p = \Delta_0 \tag{76}$$

$$m_1^{CB} + \frac{m_0^p}{1+\pi_1} = \Delta_1 \tag{77}$$

$$m_t^{CB} = \Delta_t, \text{ for } t = 2..\infty \tag{78}$$

For the irst best allocation to be a market allocation we must have (as before)

$$1 + \pi_1 = (1+r_1)^{-1} = \left(\mu A_1\tilde{K}_1^{\mu-1}L^{1-\mu}\right)^{-1} \tag{79}$$

The period 1 money market is (π_1 is defined by equality 70)

$$\frac{\Omega}{2}m_1^n = \frac{(1-\Omega)m_0^p + \frac{\Omega}{2}m_0^n}{1+\pi_1} + m_1^{CB} \tag{80}$$

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Moreover, we have the money demand for NP-agents is

$$m_0^n = (1 - \mu) A_0 \tilde{K}_0^\mu L^{-\mu} + m_0^{CB}$$

$$m_t^n = (1 - \mu) A_t \tilde{K}_t^\mu L^{-\mu} + m_1^{CB}$$

As a consequence, the period 1 money market equilibrium implies

$$\frac{\Omega}{2} \left((1 - \mu) A_1 \tilde{K}_1^\mu L^{-\mu} + m_1^{CB} \right) = \frac{(1 - \Omega) m_0^p + \frac{\Omega}{2} \left((1 - \mu) A_0 \tilde{K}_0^\mu L^{-\mu} + m_0^{CB} \right)}{1 + \pi_1} + m_1^{CB} \quad (81)$$

Trying to solve for m_0^p, we find

$$\frac{\Omega}{2} \left((1 - \mu) A_1 \tilde{K}_1^\mu L^{-\mu} + \Delta_1 \right) = \frac{\Omega}{2} \frac{(1 - \mu) A_0 \tilde{K}_0^\mu L^{-\mu}}{1 + \pi_1} + \frac{\Omega}{2} \frac{\Delta_0}{1 + \pi_1} + \Delta_1$$

In the previous equation, the terms in m_0^p now cancel out, because all agents can undo the money transfer. The period 1 money market does not depend on the value of m_0^p when the Central Bank targets a given transfer to P-agents. The previous equality does not hold in the general case, because it relates deep parameters of the model. In this case, the ZLB does directly limit the set of feasible equilibrium allocation. One can show that this condition is not fulfilled in the numerical case presented in Section 3.

H   Summary of the General Model

We provide the whole general model, the expression of the steady state of the model and the summary of parameter values chosen in the calibration.

H.1   Full Model

To ease the understanding of the general model we now provide the whole set of equations for the equilibrium under consideration. The equations concerning $NM-$households are

$$c_t^{NM} = \delta^{NM} - \tau_t$$

The equations concerning $N-$households are

$$u' \left(c_t^{NE} \right) = \beta E_t \left[\alpha u' \left(c_{t+1}^{NE} \right) + (1 - \alpha) u' \left(c_{t+1}^{NU} \right) \right] \frac{1}{1 + \pi_{t+1}}$$

$$c_t^{NU} = \frac{1}{1 + \pi_t} \frac{1 - \rho}{\alpha} m_t^{NE} + \delta^N - \tau_t$$

$$c_t^{NE} + \frac{1}{\alpha^e} m_{t+1}^{NE} = \frac{m_t^{NE}}{1 + \pi_t} + w_t - \tau_t$$

The equations concerning $N-$households are

$$u'\left(c_t^P\right) = \beta E_t \left(1 + r_{t+1}\right) u'\left(c_{t+1}^P\right)$$

$$u'\left(c_t^P\right) = \beta E_t \frac{1 + i_t}{1 + \pi_{t+1}} u'\left(c_{t+1}^P\right)$$

$$u'\left(c_t^P\right) = \beta E_t \left[\alpha^f u'\left(c_{t+1}^P\right) + \left(1 - \alpha^f\right)\left(n u'\left(c_{t+1}^{PE}\right) + (1 - n) u'\left(c_{t+1}^{PU}\right)\right)\right] \frac{1}{1 + \pi_{t+1}}$$

$$c_t^P + k_{t+1}^P + b_{t+1}^P + \frac{1}{\alpha^f} m_{t+1}^P = n w_t^P + (1 - n)\delta - \tau_t + \frac{m_t^P}{1 + \pi_t} + \left(1 + r_t\right) k_t^P + \frac{1 + i_{t-1}}{1 + \pi_t} b_t^P$$

$$c_t^{PE} = \kappa w_t - \tau_t + \frac{1}{1 + \pi_t} \frac{1 - \rho^f}{\alpha^f} m_t^P$$

$$c_t^{PU} = \kappa \delta^N - \tau_t + \frac{1}{1 + \pi_t} \frac{1 - \rho^f}{\alpha^f} m_t^P$$

First order condition of the firm

$$w_t = (1 - \mu) A_t K_t^\mu L^\mu$$

$$r_t + \lambda = \mu A_t K_t^{\mu - 1} L^{1 - \mu}$$

Budget of the State and fiscal rule

$$\frac{1 + i_{t-1}}{1 + \pi_t} b_t^o + g_t = b_{t+1}^o + m_t^{CB} + \tau_t$$

$$\tau_t = \frac{1 + i_{t-1}}{1 + \pi_t}\left(\bar{b} - m_{t-1}^{CB}\right) - \bar{b} + g$$

Law of motion of money and market equilibria

$$m_{t+1}^{tot} = \frac{m_t^{tot}}{1 + \pi_t} + m_t^{CB}$$

$$m_t^{tot} = \frac{\Omega^N}{\alpha} n m_t^{NE} + \frac{\Omega^P}{\alpha^P} n^P m_t^P$$

$$\Omega^P n^P b_t^P = b_t^o$$

$$\Omega^P n^P k_t^P = K_t$$

$$\Omega^P n^P c_t^P + \Omega^P n^{PE} c_t^{PE}$$

$$+ \Omega^P n^{PU} c_t^{PU} + \Omega^N n c_t^{NE} + \Omega^N (1 - n) c_t^{NU} + \Omega^{NM} c_t^{NM}$$

$$+ K_{t+1} = A_t K_t^\alpha L^{1-\alpha} + (1 - \delta) K_t + \Omega^{NM}\delta^{NM} + (1-n)\Omega^N\delta^N + (1-n)\Omega^P\kappa\delta$$

$$L = n\left(\Omega^N + \Omega^P \kappa\right)$$

Technology shock

$$A_t = e^{a_t}, \text{where } a_t^a = \rho^a a_{t-1} + \varepsilon_t^a$$

H.2   Steady state

The steady state allocation is easy to provide in this environment, when $\pi = 0$. In this case, $m^{CB} = \bar{b} = g = 0$. Hence $\tau = \pi = b^p = 0$. We have

$$L = n\left(\Omega^N + \Omega^P \kappa\right)$$

$$K = L\left(\frac{\frac{1}{\beta} - 1 + \lambda}{\mu}\right)^{\frac{1}{\mu-1}}$$

$$w = (1 - \mu) K^\mu L^\mu$$

$$k^P = \frac{K}{(1 - \Omega) n^p}$$

$$b^p = 0$$

From Euler equations, and budget constraint, one can obtain

$$\frac{c^{NU}}{c^{NE}} = \left(\frac{\frac{1}{\beta} - \alpha}{1 - \alpha}\right)^{-\frac{1}{\sigma}}$$

$$c^{NU} = \frac{1 - \rho}{\alpha} m^E + \delta$$

$$c^{NE} = w + \left(1 - \frac{1}{\alpha}\right) m^E$$

$$\frac{\frac{1-\rho}{\alpha} m^E + \delta}{w + \left(1 - \frac{1}{\alpha}\right) m^E} = \left(\frac{\frac{1}{\beta} - \alpha}{1 - \alpha}\right)^{-\frac{1}{\sigma}} \tag{82}$$

Thus

$$m^E = \frac{\left(\frac{\frac{1}{\beta} - \alpha}{1 - \alpha}\right)^{-\frac{1}{\sigma}} w - \delta n}{\frac{1-\rho}{\alpha} - \left(1 - \frac{1}{\alpha}\right)\left(\frac{\frac{1}{\beta} - \alpha}{1 - \alpha}\right)^{-\frac{1}{\sigma}}}$$

For $P-$households, the budget constraints give

$$c^{PE} = \kappa w + \frac{1 - \rho^f}{\alpha^f} m^P$$

$$c^{PU} = \delta + \frac{1 - \rho^f}{\alpha^f} m^P$$

$$c^P + k^P + \frac{1}{\alpha^P} m^P = n\kappa w + (1 - n)\delta + (1 + r) k^P + m^P$$

The Euler equation pf $P-$households give

$$\frac{\frac{1}{\beta} - \alpha^f}{1 - \alpha^f} = n\left(\frac{c^{PE}}{c^P}\right)^{-\sigma} + (1 - n)\left(\frac{c^{PU}}{c^P}\right)^{-\sigma}$$

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As a consequence, m^P solves

$$\frac{\frac{1}{\beta} - \alpha^f}{1 - \alpha^f} \left(n\kappa w + (1-n)\kappa\delta^N + rk^P - \left(\frac{1}{\alpha^f} - 1\right)m^P \right)^{\sigma}$$

$$= n\left(\kappa w + \frac{1 - \rho^f}{\alpha^f}m^P\right)^{-\sigma} + (1-n)\left(\kappa\delta^N + \frac{1 - \rho^f}{\alpha^f}m^P\right)^{-\sigma}$$

I   Proof of the Proposition 7

The consumption of PE and PU households is $c_t^{PE} = w_t - \tau_t + \frac{m_t^{PE}}{1+\pi_t}$ and $c_t^{PU} = \delta - \tau_t + \frac{m_t^{PU}}{1+\pi_t}$. As a consequence, $c_t^{PE} - c_t^{PU} = w_t - \delta \neq 0$. In the first best allocation we have $\tilde{c}_t^{PE} = \tilde{c}_t^{PU} = \tilde{c}_t^{P}$, what concludes the proof.